

Quantum Measurement A Coherent Description

Bas Janssens

$$\begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}$$

Quantum Measurement A Coherent Description

Master's thesis in mathematical physics
Radboud University Nijmegen

Author: Bas Janssens
Supervisor: Hans Maassen

December 2004

‘Is it your opinion, Winston, that the past has real existence?’

George Orwell
Nineteen Eighty-Four

Prologue

Quantum mechanics is weird. I had never realized this until the spring of 2003, when dr. Maassen clearly and carefully explained to me, in a two-hour lecture, why quantum mechanics cannot be a simple hidden variables theory.

This was something of a shock to me. But as the shock subsided, I realized that weirdness and lack of objective determinism were the least of my problems. There is the horrible threat of inconsistency!

Quantum mechanics is intrinsically probabilistic. Observation however involves a single outcome. In order to handle this consistently, quantum systems *must* exhibit a so-called collapse of the wave function. There are loads of specific theoretical models which show collapse on a measured system. But is this really *necessary*? Or would it be possible to perform a measurement without collapse?

Let's kill the tension right away. The answers are yes and no respectively. If you bear with me for 60 short and exciting pages, I'll prove to you that transfer of information out of a system always causes collapse on that system. Along the way, we will gain quantitative insight into the balance between information gain and state disturbance.

Well then. Now that I've skillfully tricked you into reading the rest of this thesis, I am left with the pleasant task of thanking people. It goes without saying that I am grateful to my friends and family for such diverse matters as giving birth or money (which only goes for a fairly restricted class of family members) and tolerating or even supporting my (rather obnoxious) enthusiasm. (The latter goes for anyone having had even remote contact with me over the past year.)

But here and now, I would like to express my gratitude to those who made a direct contribution to this thesis: Prof. Ronald Kleiss for agreeing to be my official first and unofficial second supervisor. Prof. Klaas Landsman for a careful reading of this text, and for many useful suggestions. Mădălin Guță for suggesting a simple proof of lemma (11) in the case of completely positive maps, putting me on track for proposition (25). Janneke Blokland for useful advice on the editing. And most of all, I would like to thank dr. Hans Maassen.

The two-hour lecture I told you about was part of his course in ‘*Quantum Probability, Quantum Information Theory and Quantum Computing*’ which lies at the very heart of this thesis. Indeed, insiders will recognize chapter 3 as a mere extension of the lecture notes [Maa]. I am thankful for the excellent guidance and for the countless conversations we had, shaping my perception of quantum mechanics into its present state. They were sometimes slightly confusing, but always pleasant and fertile. I have learnt much from Hans over the past year, and I would be proud if his style may be seen, reflected in my writing.

Bas Janssens
December 2004

Definitions and Conventions

\mathbb{P}	Generically denotes a classical probability distribution.
$\mathbb{E}_{\mathbb{P}}(\mathfrak{a})$	$\mathbb{E}_{\mathbb{P}}(\mathfrak{a}) = \int_{\Omega} \mathfrak{a}(\omega) \mathbb{P}(d\omega)$, the expectation of random variable \mathfrak{a} under \mathbb{P} .
$\mathbf{Var}_{\mathbb{P}}(\mathfrak{a})$	$\mathbf{Var}_{\mathbb{P}}(\mathfrak{a}) = \mathbb{E}_{\mathbb{P}}(\mathfrak{a}^2) - \mathbb{E}_{\mathbb{P}}(\mathfrak{a})^2$, the variance under \mathbb{P} of random variable \mathfrak{a} .
$\mathbf{Cov}_{\mathbb{P}}(\mathfrak{a}, \mathfrak{b})$	$\mathbf{Cov}_{\mathbb{P}}(\mathfrak{a}, \mathfrak{b}) = \mathbb{E}_{\mathbb{P}}(\mathfrak{a}\mathfrak{b}) - \mathbb{E}_{\mathbb{P}}(\mathfrak{a})\mathbb{E}_{\mathbb{P}}(\mathfrak{b})$, the covariance under \mathbb{P} of \mathfrak{a} and \mathfrak{b} .
$\mathcal{C}(V)$	If $V \subset \mathbb{C}$, then $\mathcal{C}(V)$ is the space of continuous functions on V .
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$	Script letters denote C*-algebras.
α	Automorphisms are usually denoted by α . By an automorphism of a C*-algebra, we <i>always</i> mean a C*-automorphism, i.e. we assume $\alpha(A)^{\dagger} = \alpha(A^{\dagger})$.
$\mathcal{C}(A)$	If $A \in \mathcal{A}$, then $\mathcal{C}(A) \subset \mathcal{A}$ is the C*-sub-algebra generated by A and \mathbb{I} .
$\mathcal{S}(\mathcal{A})$	Denotes the convex <i>state space</i> of normalized positive linear functionals $\mathcal{A} \rightarrow \mathbb{C}$.
\bar{z}	The complex conjugate of a complex number $z \in \mathbb{C}$.
A^{\dagger}	The Hermitean conjugate of $A \in \mathcal{A}$.
$\Re A$	$\Re A = \frac{1}{2}(A + A^{\dagger})$: the real part of $A \in \mathcal{A}$.
$\Im A$	$\Im A = \frac{1}{2i}(A - A^{\dagger})$: the imaginary part of $A \in \mathcal{A}$.
\mathbf{M}^*	If \mathbf{M} is a positive linear mapping $\mathcal{B} \rightarrow \mathcal{A}$, then its dual \mathbf{M}^* is a $\mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{B})$ mapping defined by $\mathbf{M}^*(\rho) = \rho \circ \mathbf{M}$.
$\mathbf{Var}_{\rho}(A)$	$\mathbf{Var}_{\rho}(A) = \rho(A^{\dagger}A) - \overline{\rho(A)}\rho(A)$, the variance under ρ of $A \in \mathcal{A}$.
$\mathbf{Cov}_{\rho}(A, B)$	$\mathbf{Cov}_{\rho}(A, B) = \rho(A^{\dagger}B) - \overline{\rho(A)}\rho(B)$, the covariance under ρ of $A, B \in \mathcal{A}$.
$\mathbf{Spec}(A)$	The spectrum of $A \in \mathcal{A}$.
Y'	$Y' = \{ A \in \mathcal{A} \mid [A, Y] = 0 \}$, the relative commutant of Y .

M_n	The algebra of $n \times n$ -matrices acting on \mathbb{C}^n .
ψ_+	$\psi_+ = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$ in \mathbb{C}_2
ψ_-	$\psi_- = \begin{pmatrix} & 0 \\ 1 & \end{pmatrix}$ in \mathbb{C}_2
σ_x	$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in M_2 .
σ_y	$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ in M_2 .
σ_z	$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in M_2 .
\mathbf{P}_+	$\mathbf{P}_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in M_2 .
\mathbf{P}_-	$\mathbf{P}_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in M_2 .

Contents

Prologue	i
Definitions and Conventions	iii
Introduction	vii
1 Quantum Measurement	1
1.1 Postulates of Quantum Mechanics	1
1.2 Interpretation of the Postulates	3
1.2.1 An Inconsistent Interpretation	3
1.2.2 A Traditional Interpretation	4
1.2.3 My Favourite Interpretation	6
1.3 Measurement	10
1.3.1 Automorphic Measurement	10
1.3.2 More General Measurement	10
1.4 State Reduction	12
1.4.1 An Example of Reduction	12
1.4.2 Reduction as a Consequence of Measurement	13
1.4.3 Imperfect Reduction after Imperfect Measurement	16
1.5 State Collapse	20
1.5.1 Perfect Collapse after Perfect Measurement	20
1.5.2 Imperfect Collapse after Imperfect Measurement	21
2 Macroscopic Observables	25
2.1 Collapse on the Measurement Apparatus	26
2.2 Collapse on Local and Global Observables	27
2.2.1 K. Hepp: Quasiloca1 Algebras	27
2.2.2 Local Algebras	29
2.3 Global Information Leakage	32
2.3.1 Information Leakage After Measurement	32
2.3.2 Information Leakage in General	32
2.4 Conclusion	34

3	Measurement Inequalities	35
3.1	Completely Positive Operations	35
3.2	A Cauchy-Schwarz Inequality	37
3.3	Quantum Measurement	42
3.3.1	Introduction	42
3.3.2	Examples of Unbiased Measurement	44
3.3.3	Structure of a Perfect Measurement	45
3.3.4	Simultaneous Measurement	47
3.4	The Heisenberg Principle	49
3.4.1	Global Generalization	49
3.4.2	Local Generalization	52
3.5	State Reduction and Collapse	53
3.5.1	State Reduction	53
3.5.2	State Collapse	54
3.5.3	Generalized State Collapse	55
3.5.4	Generalized State Reduction	58
3.6	A Paradox Resolved	60
	Epilogue	61

Introduction

In the prologue, I already made a brief sketch of the subject of this thesis. Allow me to add a few details.

This is a Master's thesis in mathematical physics, written in the period September 2003 – October 2004 at the Radboud University Nijmegen, under supervision of dr. Hans Maassen. Its aim is twofold:

- First of all, I intend to prove general theorems, showing that state collapse on a measured system is a necessary consequence of transporting information out of that system.
- Afterwards, we shall investigate the balance between information gain and state disturbance in a more quantitative way.

The first point is in contrast with authors like Joos, Zeh and Zurek who, if I understand correctly, endeavour to find specific models of decoherence on a system, independent of the information transfer.

Another group of authors (Hepp, Lieb, Sewell, Rieckers) transports information to a central pointer in an infinite system. Although centrality of the pointer enables them to model a global collapse on all observables, it inhibits them from using automorphic time evolution.

I on the other hand will use finite (not necessarily finite dimensional) systems, transporting information to non-central pointers. I do not restrict attention to automorphic time evolution, but it is allowed as a special case in each proposition in this thesis. We will see that collapse of the wave function then automatically occurs on the examined system. But we will also show that an approximate collapse occurs on a much wider range of observables, including the observables of the measurement apparatus.

This brings us to the similarities with this second group of authors. Most of chapter 2 is based on a most original idea, due to Hepp, that collapse has to do with the difference in size between the pointer (macroscopic) and the observable on which collapse is supposed to occur (microscopic). In fact, the whole point of using finite systems was originally just to get a quantitative estimate of how the idealization of infinite systems is reached in the realistic case of a large but finite system. Exactly how large must the system be? What observables defy collapse? You will find answers in chapter 2.

But before entering the bulk of this thesis, I would like to caution the reader about two points which might seem essential at first sight, but are in fact merely a matter of personal preference of the author:

- In the postulates of quantum mechanics, systems are modelled by C^* -algebras. This is not essential: I might just as well have chosen von Neumann algebras. If you are not familiar with operator algebra techniques altogether, you may take in mind $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on the Hilbert space \mathcal{H} . This example will serve you well throughout the text.
- I also wield a rather unorthodox interpretation of quantum mechanics. I do this simply because it is in my eyes the simplest possible interpretation. Do not be distracted: the issue of interpreting quantum mechanics is quite separate from the issue of state collapse after information transfer. If you do not like my interpretation of quantum mechanics, simply take your own favourite interpretation and apply it to the mathematics in this thesis. The result will probably be satisfactory.

Finally, a short note on source material. This thesis contains a grand total of 26 lemmas and propositions, plus another 10 corollaries. Of course not all of these are new. There are three possibilities.

Sometimes, I prove theorems already proven by others before. In that case of course, I refer to this person explicitly. I have also formulated a number of results which have been widely known for a long time. In that case I explicitly mention that it is a ‘standard result’. This leaves a total of 19 lemmas and propositions plus 7 corollaries that are neither attributed to one particular person nor explicitly labelled ‘standard result’. These are of my own invention. The reader will understand however that there exist no guarantees that no one else has invented them before. If so, I have not been able to track this down.

Now, without more ado, we finally move from the disclaimer to the actual physics. Enjoy the ride...

Chapter 1

Quantum Measurement

In order to investigate quantum measurement, we dwell on the foundations of quantum mechanics for a short while.

1.1 Postulates of Quantum Mechanics

Regardless of their interpretation, we will postulate the existence of the three mathematical protagonists of quantum theory: an algebra, a state and a one-parameter group of automorphisms.

Postulate 1 *A quantum mechanical system will be modelled mathematically by a unital C^* -algebra \mathcal{A} , the algebra of observables.*

Quite often, $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on some Hilbert space \mathcal{H} . If you're not familiar with C^* -algebras, this is a good example to keep in mind. In general, any C^* -algebra \mathcal{A} has a faithful representation on some Hilbert space \mathcal{H} , see [K&R, p. 281].

Postulate 2 *A physical state of this system will be modelled mathematically by a (normalized) positive linear functional ρ on \mathcal{A} .*

The set of all possible states on \mathcal{A} makes up state space, $\mathcal{S}(\mathcal{A})$. A unit vector $|\psi\rangle \in \mathcal{H}$, for example, induces a state ρ on $\mathcal{B}(\mathcal{H})$ by $\rho(A) \stackrel{\text{def}}{=} \langle\psi|A|\psi\rangle$. Respecting conventional abuse of language instead of common sense, we will not always distinguish between vector states and vectors.

But these are not the only states allowed for the system. If for each positive integer i , we have a normalized vector $|\psi_i\rangle \in \mathcal{H}$ and a number $p_i \in [0, 1]$ such that $\sum_{i=1}^{\infty} p_i = 1$, we may form the state $\rho(A) \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} p_i \langle\psi_i|A|\psi_i\rangle$ on $\mathcal{B}(\mathcal{H})$. This is a positive linear functional on \mathcal{A} as well, and thus perfectly acceptable as a physical state. This particular state happens to be normal, i.e. continuous in the weak operator topology. But we also accept non-normal functionals as states, according to the postulate above.

Postulate 3 *Time evolution in an isolated system is modelled mathematically by a one-parameter group of automorphisms of \mathcal{A} : $t \mapsto \alpha_t$. That is, $\alpha_{t+s} = \alpha_t \circ \alpha_s$ for all $t, s \in \mathbb{R}$.*

Let $\alpha^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A})$ denote the dual action of α on state space: $\alpha^*(\rho) \stackrel{\text{def}}{=} \rho \circ \alpha$. Then a state ρ on time t_0 will evolve to a state $\alpha^*_{(t_1-t_0)}\rho$ on time t_1 .

A unitary $U \in \mathcal{A}$ induces an automorphism α of \mathcal{A} by $\alpha(A) = U^\dagger A U$. Usually the one-parameter group of automorphisms describing time-evolution is induced by a one-parameter group of unitaries $t \mapsto U_t \in \mathcal{A}$. So after a time t the observable A will evolve to $\alpha_t(A) = U_t^\dagger A U_t$. Translating to the Schrödinger picture, a state ρ will evolve to ρ_t : $\rho_t(A) = \alpha_t^* \rho(A) = \rho(U_t^\dagger A U_t)$. If ρ is the vector state induced by $|\psi\rangle$, then it evolves to $\rho_t(A) = \langle \psi | U_t^\dagger A U_t | \psi \rangle$. In other words, ρ_t is the vector state induced by $U_t |\psi\rangle$.

Induced Probability Measures

States on a C^* -algebra have the pleasant property of inducing probability measures. This is clear from the following standard result:

Proposition 1 (Induced Probability Measure) *Let \mathcal{A} be a C^* -algebra. Let $X \in \mathcal{A}$ be Hermitean. Then each state $\rho \in \mathcal{S}(\mathcal{A})$ defines uniquely a probability measure $\mathbb{P}_{\rho,X}$ on the Borel σ -algebra of $\mathbf{Spec}(X)$ such that $\rho(f(X)) = \int f(x) \mathbb{P}_{\rho,X}(dx)$ for all $f \in \mathcal{C}(\mathbf{Spec}(X))$.*

Proof:

By the Gel'fand transform (see e.g. [K&R, p. 271]), we have an injective C^* -homomorphism $f \mapsto f(X)$ of $\mathcal{C}(\mathbf{Spec}(X))$, the continuous functions on the spectrum of X , into \mathcal{A} . We compose this with the state $\rho : \mathcal{A} \rightarrow \mathbb{C}$ to yield a functional \mathbb{E} on $\mathcal{C}(\mathbf{Spec}(X))$. In short, $\mathbb{E}(f) \stackrel{\text{def}}{=} \rho(f(X))$. \mathbb{E} is positive: if $f \geq 0$, then $f(X) \geq 0$ in the operator ordering, hence $\mathbb{E}(f) = \rho(f(X)) \geq 0$ since ρ is a positive functional.

By the Riesz representation theorem (see [Coh, p. 209]), \mathbb{E} defines a unique Borel measure $\mathbb{P}_{\rho,X}$ on $\mathbf{Spec}(X)$ satisfying $\mathbb{E}(f) = \int f(x) \mathbb{P}_{\rho,X}(dx)$. That this is a probability measure indeed can be seen from $\mathbb{P}_{\rho,X}(\mathbf{Spec}(X)) = \rho(\mathbb{I}) = 1$.

q.e.d.

For example, consider the physically relevant case of a normal (i.e. weakly continuous) state ρ on a von Neumann algebra \mathcal{A} . Then X defines¹ a projection valued measure $V \mapsto \mathbf{P}(V)$. In this setting, $\mathbb{P}_{\rho,X}$ is simply defined by $\mathbb{P}_{\rho,X}(V) \stackrel{\text{def}}{=} \rho(\mathbf{P}(V))$. In case ρ is a vector state $|\psi\rangle$ and X has discrete spectrum with non-degenerate eigenspaces, $X = \sum_i x_i |\psi_i\rangle\langle\psi_i|$, this amounts to $\mathbb{P}_{\rho,X}(\{x_i\}) = |\langle\psi_i|\psi\rangle|^2$.

¹V is a Borel subset of $\mathbf{Spec}(X)$. In this thesis, all subsets of spectra will be assumed Borel measurable.

1.2 Interpretation of the Postulates

Up until now, we only have the existence of mathematical objects. In order to link mathematics to physical experiment, we seek an interpretation of the postulates above.

1.2.1 An Inconsistent Interpretation

Proposition (1) cries out for an interpretation of postulates (1) and (2). The first that comes to mind would be:

Interpretation 1

- *A quantum mechanical system will be represented by a unital C^* -algebra \mathcal{A} , the algebra of observables.*
- *Each random variable \mathfrak{a} is represented by a Hermitean $A \in \mathcal{A}$.*
- *The random variable \mathfrak{a} objectively takes values in $\mathbf{Spec}(A)$. If the system is in state $\rho \in \mathcal{S}(\mathcal{A})$, then the probability that \mathfrak{a} takes value in V is $\mathbb{P}_{\rho,A}(V)$.*

This interpretation allows us to interpret the state $\rho(A) = \sum_{i=1}^{\infty} p_i \langle \psi_i | A | \psi_i \rangle$ as a system in state $|\psi_i\rangle$ with probability² p_i .

Unfortunately, the interpretation above is inconsistent, at least for $\mathcal{A} = M_2 \otimes M_2$. For each Borel subset $V \subset \mathbf{Spec}(A)$, $\mathbb{P}_{\rho,A}(V)$ gives the probability that \mathfrak{a} takes value in V . Therefore \mathfrak{a} is a random variable on the probability space $(\mathbf{Spec}(A), \mathbb{P}_{\rho,A}(V))$. Similarly, \mathfrak{b} is a random variable on the probability space $(\mathbf{Spec}(B), \mathbb{P}_{\rho,B}(V))$.

But if \mathfrak{a} and \mathfrak{b} both take objective values, then there must exist *some* probability distribution \mathbb{P} on $\mathbf{Spec}(A) \times \mathbf{Spec}(B)$ such that $\mathbb{P}(V \times W)$ is the probability that \mathfrak{a} lies in V and \mathfrak{b} in W . So \mathfrak{a} and \mathfrak{b} must be random variables on *the same* probability space $(\mathbf{Spec}(A) \times \mathbf{Spec}(B), \mathbb{P})$, having $\mathbb{P}_{\rho,A}$ and $\mathbb{P}_{\rho,B}$ as marginal probability distributions.

This means that each set of random variables has to satisfy Bell's inequalities. (See [Jau, p. 116] for a very thorough and [B&J, p. 673] for a very accessible version.) But in $M_2 \otimes M_2$, Bell's inequalities are violated for certain choices of ρ . As a result, interpretation (1) is inconsistent.

This is not exclusively the case for $\mathcal{A} = M_2 \otimes M_2$. Interpretation (1) is inconsistent for $\mathcal{A} = \mathcal{B}(\mathcal{H})$ with $\dim(\mathcal{H}) > 2$ (see [K&S]).

Induced Joint Probability Measures

In order to pave the way for a consistent interpretation, we will extend proposition (1) with the following standard result:

²This interpretation of ρ is slightly less straightforward than it seems at first sight, since the decomposition of ρ into pure states may not be unique.

Proposition 2 (Induced Joint Probability Measures) *Let $X, Y \in \mathcal{A}$ be Hermitean such that $[X, Y] = 0$. Then each state $\rho \in \mathcal{S}(\mathcal{A})$ defines uniquely a probability measure $\mathbb{P}_{\rho, X, Y}$ on the Borel σ -algebra of $\mathbf{Spec}(X) \times \mathbf{Spec}(Y)$ such that $\rho(f(X)g(Y)) = \int f(x)g(y)\mathbb{P}_{\rho, X, Y}(dx, dy)$. In particular:*

$$\mathbb{E}_{\mathbb{P}}(X) = \rho(X), \quad \mathbb{E}_{\mathbb{P}}(Y) = \rho(Y)$$

$$\mathbf{cov}_{\mathbb{P}}(X, Y) = \rho(XY) - \rho(X)\rho(Y)$$

Proof

For continuous f and g on the spectra of X and Y , we have once again $f(X)$ and $g(Y)$ by the Gel'fand transform (see [K&R, p. 271]). We define a functional \mathbb{E} on $\mathcal{C}(\mathbf{Spec}(X) \times \mathbf{Spec}(Y)) = \mathcal{C}(\mathbf{Spec}(X)) \otimes \mathcal{C}(\mathbf{Spec}(Y))$ by $\mathbb{E}(f \otimes g) \stackrel{\text{def}}{=} \rho(f(X)g(Y))$. \mathbb{E} is positive: if $f \otimes g \geq 0$, choose $f, g \geq 0$. Then, because $[f(X), g(Y)] = 0$, $f(X)g(Y) = \sqrt{f(X)}g(Y)\sqrt{f(X)} \geq 0$ in the operator ordering. Now since ρ is a positive functional, $\mathbb{E}(f \otimes g) = \rho(\sqrt{f(X)}g(Y)\sqrt{f(X)}) \geq 0$.

By the Riesz representation theorem (see [Coh, p. 209]), \mathbb{E} defines a unique Borel measure $\mathbb{P}_{\rho, X, Y}$ on $\mathbf{Spec}(X) \times \mathbf{Spec}(Y)$ satisfying $\mathbb{E}(f \otimes g) = \int f(x)g(y)\mathbb{P}_{\rho, X, Y}(dx, dy)$. Of course $\mathbb{P}_{\rho, X, Y}(\mathbf{Spec}(X) \times \mathbf{Spec}(Y)) = \rho(\mathbb{I}) = 1$.

q. e. d.

For example, let ρ be a normal (i.e. weakly continuous) state on a von Neumann algebra \mathcal{A} . Then X and Y define commuting projection valued measures $V \mapsto \mathbf{P}(V)$ and $W \mapsto \mathbf{Q}(W)$. In this setting, $\mathbb{P}_{\rho, X, Y}$ is simply defined by the formula $\mathbb{P}_{\rho, X, Y}(V \times W) \stackrel{\text{def}}{=} \rho(\mathbf{P}(V)\mathbf{Q}(W))$.

1.2.2 A Traditional Interpretation

A standard interpretation of postulates (1) through (3) is the following:

Interpretation 2

- *A quantum mechanical system will be represented by a unital C^* -algebra \mathcal{A} , the algebra of observables³.*
- *At any fixed time, there is one state $\rho \in \mathcal{S}(\mathcal{A})$, representing all knowledge concerning \mathcal{A} .*
- *Each observable is represented by a Hermitean element A of \mathcal{A} ⁴.*
- *There is an action called ‘measurement’. Observables only take on objective values if they are measured. Joint measurement of commuting observables A and B yields values of A in $V \subset \mathbf{Spec}(A)$ and of B in $W \subset \mathbf{Spec}(B)$ with probability $\mathbb{P}_{\rho, A, B}(V \times W)$.*

³In this case, we will denote both system and algebra by \mathcal{A} .

⁴Again, both the observable and the Hermitean element will be referred to by A .

- *Time evolution on an undisturbed system \mathcal{A} is represented by a one-parameter group of automorphisms of \mathcal{A} : $t \mapsto \alpha_t$. A state ρ at time t_0 will evolve to the state $\alpha_{(t_1-t_0)}^* \rho$ at time t_1 .*

Bell's inequalities do not apply here because it is not possible to perform a simultaneous measurement on non-commuting observables. One problem solved.

In order to interpret a state ρ on a system \mathcal{A} , an outside observer is introduced, performing this abstract 'measurement of A '. This has the effect of forcing A to take on an objective value. Neither the observer, nor the measurement are described within the framework of quantum mechanics. But they do have a physically observable effect on the system.

State Reduction

We will demonstrate this with a simple example. Let \mathcal{A} be M_2 , the algebra of 2×2 -matrices acting on $\mathcal{H} = \mathbb{C}^2$. This describes an electron, having only spin-properties. Let the observable A be $\sigma_z \in M_2$, the spin in the z -direction⁵. σ_z has spectrum $\mathbf{Spec}(\sigma_z) = \{1, -1\}$. Suppose that σ_z is measured. According to interpretation (2), either $\sigma_z = 1$ or $\sigma_z = -1$. (With probabilities $\mathbb{P}_{\rho, \sigma_z}(\{1\})$ and $\mathbb{P}_{\rho, \sigma_z}(\{-1\})$ respectively.)

Suppose that the measurement is *repeatable*. This means that a second measurement of σ_z , performed immediately after the first, would yield the same result. Then after measurement, knowledge of the system has increased: if the measurement has revealed $\sigma_z = 1$, then we know that any future measurement of σ_z will yield $\sigma_z = 1$ again. According to interpretation (2), we must now describe the system by a different mathematical state than before, one yielding $\sigma_z = 1$ with certainty. The *only* state on \mathcal{A} which does this is the vector state $|\psi_+\rangle$. This change of state forced by measurement is called *state reduction*.

Classical State Reduction

In classical probability theory measurement is also possible, and the reduction it produces is called 'conditioning'. A classical spin-system is described by a probability distribution \mathbb{P} on a classical probability space $\Omega = \{+1, -1\}$. Repeatable measurement can be performed on the random variable $\sigma_z : \Omega \rightarrow \mathbb{R}$ defined by $\sigma_z(\omega) = \omega$. If $\sigma_z = 1$, the observer will update \mathbb{P} to the conditioned probability distribution $\mathbb{P}(\bullet | [\sigma_z = 1])$ defined by

$$\mathbb{P}(V | [\sigma_z = 1]) \stackrel{\text{def}}{=} \frac{\mathbb{P}(V \cap [\sigma_z = 1])}{\mathbb{P}([\sigma_z = 1])} = \delta_{+1}(V) \quad (1.1)$$

where δ_{+1} is the point measure on $\omega = +1$. Another observer, unaware of the measurement outcome, will describe the system by a distribution $\mathbb{P}' = \sum_{x=\pm 1} \mathbb{P}([\sigma_z = x]) \mathbb{P}(\bullet | [\sigma_z = x])$. This equals the original distribution \mathbb{P} . So in a classical probability space, state reduction is *subjective*. It can be attributed entirely to the increase of knowledge of the first observer.

⁵For notation on spin systems, see page iv.

Quantum State Reduction

In quantum mechanics the situation is radically different: objective collapse after measurement can be verified experimentally. Suppose M_2 is in vector state $\alpha|\psi_+\rangle + \beta|\psi_-\rangle$. Repeatable σ_z -measurement is then performed on M_2 . $\sigma_z = 1$ will occur with probability $|\alpha|^2$ and $\sigma_z = -1$ with probability $|\beta|^2$. So after measurement, the system is in state ψ_+ with probability $|\alpha|^2$ and in state ψ_- with probability $|\beta|^2$. The situation is now objectively different from the one before: whether we know the value of σ_z or not, we may perform σ_x -measurement. In both ψ_+ and ψ_- , the probability of finding $\sigma_x = 1$ equals $1/2$. But before measurement, in the state $\alpha|\psi_+\rangle + \beta|\psi_-\rangle$, this probability would have been $1/2 + \Re(\bar{\alpha}\beta)$.

So experimental verification of collapse can be achieved as follows: start with a system in state $1/\sqrt{2}|\psi_+\rangle + 1/\sqrt{2}|\psi_-\rangle$. The first observer measures σ_z , the second σ_x . This is repeated a number of times. As soon as the second observer measures $\sigma_x = -1$, the point is made. Collapse is verified objectively.

In summary, repeatable measurement of σ_z always causes the state of M_2 to jump:

- If the observer learns that $\sigma_z = +1$, the state jumps from $\alpha|\psi_+\rangle + \beta|\psi_-\rangle$ to ψ_+ . We will call this change ‘state *reduction*’
- If the observer is ignorant of the outcome, the state jumps from the vector state $\alpha|\psi_+\rangle + \beta|\psi_-\rangle$ to the mixed state $|\alpha|^2|\psi_+\rangle\langle\psi_+| + |\beta|^2|\psi_-\rangle\langle\psi_-|$. We will call this change ‘state *collapse*’

In the literature, each jump is commonly referred to as both collapse and reduction. In order to avoid confusion, we shall keep these notions separate.

Comments

There is a sharp and physically observable schism between the situation before and after measurement. Therefore, it is important to know if measurement takes place and if so, exactly when⁶. In practice, there is hardly any doubt as to when it takes place. And if you feel comfortable with interpretation (2), you may read the rest of this thesis as an attempt to explain why, in practice, the exact point in time where the actual reduction takes place is not of much importance.

But personally, I feel rather uncomfortable with the need for outside observers, not described within quantum theory, exerting influence on a system that *is* described by quantum theory. I would like my physical theory to be a universe in itself. It should describe all the observables that can be measured. But also all observers, and the act of measurement itself.

1.2.3 My Favourite Interpretation

First of all then, we want to describe all interference with a system \mathcal{A} within the framework of quantum mechanics. This does not mean that time evolution on \mathcal{A} is always

⁶J.Bell puts it like this (see [Bell]): “... so long as the wave packet reduction is an essential component, and so long as we do not know exactly when and how it takes over from the Schrödinger equation, we do not have an exact and unambiguous formulation of our most fundamental physical theory.”

automorphic. But it does mean that there is always a system $\mathcal{D} \supseteq \mathcal{A}$ such that time evolution is automorphic on \mathcal{D} . Think of \mathcal{D} as the entire universe, if you have to.

Secondly, observables never take on objective values *at all*. Physics is not about objective events. Physics intends to predict the observations made by observers. So a physical theory should have:

- A list of all observers \mathcal{C} .
- For each \mathcal{C} , a list of observables that are directly observed by \mathcal{C} .

Then it should predict the probabilities of the observations made by each separate observer without having to make any reference to other observers or objective reality.

Interpretation 3

- *There is one largest universal system. It is represented by a C^* -algebra \mathcal{D} .*
- *Each observable is represented by a Hermitean element A of \mathcal{D} . Each observer is represented by an abelian C^* -subalgebra $\mathcal{C} \subset \mathcal{D}$. \mathcal{C} directly observes all Hermitean A in \mathcal{C} . \mathcal{C} cannot directly observe A if $A \notin \mathcal{C}$.*
- *At any fixed time, there is one $\rho \in \mathcal{S}(\mathcal{D})$ representing the physical state of \mathcal{D} . Each direct observation of any $A \in \mathcal{C}$ made by \mathcal{C} has a value in $\mathbf{Spec}(A)$, the spectrum of A . If $A, B \in \mathcal{C}$, then the probability that \mathcal{C} observes a value of A in $V \subset \mathbf{Spec}(A)$ and a value of B in $W \subset \mathbf{Spec}(B)$ is given by $\mathbb{P}_{\rho, A, B}(V \times W)$.*
- *Even while observation takes place, time evolution is represented by a one-parameter group of automorphisms of \mathcal{D} : $t \mapsto \alpha_t$. A state ρ at time t_0 will evolve to the state $\alpha_{(t_1 - t_0)}^* \rho$ at time t_1 .*

All $A \in \mathcal{C}$ are observed by \mathcal{C} and all probabilities of finding joint values are given by the theory. This means that the observables in \mathcal{C} may be considered random variables on some classical probability space. If $A \in \mathcal{D}$ and $B \in \mathcal{D}$ do not commute, then they cannot be directly observed by the same observer. Both are random variables, but not on the same probability space. Therefore Bell's inequalities do not apply.

Each subsystem of \mathcal{D} is of course represented by some subalgebra $\mathcal{A} \subset \mathcal{D}$. If this subsystem happens to be invariant under the time evolution of \mathcal{D} , then we can regard \mathcal{A} as an isolated system with time evolution $\alpha_t|_{\mathcal{A}}$.

Direct and Indirect Observation

Notice that one single observer \mathcal{C} cannot directly observe all $A \in \mathcal{D}$ if \mathcal{D} is not abelian. Suppose for example that the observer is an eye. This eye observes directly the voltage on each of its neurons. Indirectly, it can also observe say a painting on the other side of the room: rays of light carry information from the painting to the retina and the eye observes voltages in the retina directly. There is a radical difference between direct and indirect observation.

Direct observation is the most primitive form. It is needed to link mathematics to experience. It is restricted to observables A in the observer \mathcal{C} , and it does not result in any objective collapse.

Indirect observation of observables outside \mathcal{C} is possible. However, this requires some pre-formed image of the outside world: the eye must trust photons to travel in straight lines. We will call this indirect observation *measurement*, and we will come to it later on.

Reduction and Collapse

Given a state ρ on \mathcal{A} , we now formally define its *reduced* state ρ_Y on \mathcal{A} :

Definition 1 (Reduced State) *Let \mathcal{A} be a C^* -algebra. Let $Y \in \mathcal{A}$, let $\rho \in \mathcal{S}(\mathcal{A})$ and let $\rho(Y^\dagger Y) \neq 0$. Then we define⁷ the state ρ_Y on \mathcal{A} by*

$$\rho_Y(A) \stackrel{\text{def}}{=} \frac{\rho(Y^\dagger A Y)}{\rho(Y^\dagger Y)}.$$

Suppose that a countable decomposition $\{V_i | i \in I\}$ of the spectrum of Y is given. This means that V_i are Borel subsets of $\mathbf{Spec}(Y)$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$, and $\bigcup_{i \in I} V_i = \mathbf{Spec}(Y)$. Then we also have a *collapsed* state $\mathbf{C}^*\rho$:

Definition 2 (Collapsed State) *Let \mathcal{A} be a von Neumann algebra. Let $Y \in \mathcal{A}$ be Hermitean, and let $\{V_i | i \in I\}$ be a countable decomposition of its spectrum. Then if $\rho \in \mathcal{S}(\mathcal{A})$, its collapsed state $\mathbf{C}^*(\rho)$ is defined by*

$$\mathbf{C}^*(\rho)(A) \stackrel{\text{def}}{=} \rho\left(\sum_I \mathbf{P}(V_i) A \mathbf{P}(V_i)\right)$$

where $V \mapsto \mathbf{P}(V)$ is the projection valued measure of Y .

Note that $\mathbf{C}^*(\rho)(A) = \sum_I \mathbb{P}_{\rho,Y}(V_i) \rho_{\mathbf{P}(V_i)}(A)$. The collapsed state is the sum of reduced states, weighed over the probability distribution.

Conditioning

Suppose that ρ is a normal state on a von Neumann algebra \mathcal{A} . Suppose \mathcal{C} directly observes both A and B so that $[A, B] = 0$. Then A has projection valued measure $V \mapsto \mathbf{P}(V)$ and B has commuting projection valued measure $W \mapsto \mathbf{Q}(W)$. One can then calculate the probability distribution of $B \in \mathcal{C}$ provided that \mathcal{C} observes a value of A in $V \subset \mathbf{Spec}(A)$:

$$\begin{aligned} \mathbb{P}_{\rho,A,B}([B \text{ in } W] | [A \text{ in } V]) &\stackrel{\text{def}}{=} \frac{\mathbb{P}_{\rho,A,B}([B \text{ in } W \text{ and } A \text{ in } V])}{\mathbb{P}_{\rho,A,B}([A \text{ in } V])} \\ &= \frac{\rho(\mathbf{P}(V) \mathbf{Q}(W))}{\rho(\mathbf{P}(V))} \\ &= \frac{\rho(\mathbf{P}(V) \mathbf{Q}(W) \mathbf{P}(V))}{\rho(\mathbf{P}(V))} \\ &= \rho_{\mathbf{P}(V)}(\mathbf{Q}(W)) \\ &= \mathbb{P}_{\rho_{\mathbf{P}(V)},B}([B \text{ in } W]). \end{aligned}$$

⁷We will see that if $\rho(Y^\dagger Y) = 0$, there will never be any need for a reduced state. From now on, when mentioning reduced states, I will tacitly assume their existence. Even in theorems.

The reduced state $\rho_{\mathbf{P}(V)}$ induces the conditional probability measure on any $B \in A'$, i.e. any B such that $[B, A] = 0$. We thus have an interpretation of $\rho_{\mathbf{P}(V)}$ considered as a state on A' .

A Benevolent Word of Caution to the Reader

Since this is my thesis, I will proceed with my favourite interpretation. But keep in mind that this is merely a way of interpreting the mathematics to come. As such, theorems are universal and do not hinge on any interpretation.

1.3 Measurement

We have stated that an observer \mathcal{C} can observe $X \in \mathcal{D}$ indirectly, even if $X \notin \mathcal{C}$. This is accomplished by transferring information from X to some $Y \in \mathcal{C}$, the so-called ‘pointer observable’, and then observing Y directly. If \mathcal{D} is in state ρ , time evolution α_t must be such that $\mathbb{P}_{\alpha_t^*(\rho), Y} = \mathbb{P}_{\rho, X}$: the probability distribution that \mathcal{C} finds when observing Y at time t exactly equals the one any $\mathcal{C} \ni X$ would find when observing X at time 0.

Since we only need the automorphism α_t at the fixed time t when measurement is completed, we will drop the suffix t from now on.

1.3.1 Automorphic Measurement

If α is such that $\alpha(Y) = X$, then $\mathbb{P}_{\alpha^*(\rho), Y} = \mathbb{P}_{\rho, X}$ for all $\rho \in \mathcal{S}(\mathcal{D})$. It is immediately clear that the averages are the same, $\alpha^*(\rho)(Y) = \rho(\alpha(Y)) = \rho(X)$. But since α is an automorphism, $\alpha(f(A)) = f(\alpha(A))$ for all $A \in \mathcal{D}$ and $f \in \mathcal{C}(\mathbf{Spec}(A))$. Therefore the expectation values \mathbb{E} used in the proof of proposition (1) to construct the probability measures are automatically identical: $\alpha^*\rho(f(Y)) = \rho(f(X))$. Then the probability distributions themselves must be identical.

Example

Let $\mathcal{D} = M_2 \otimes M_2$. Think of $M_2 \otimes \mathbb{I}$ as an electron with spin. $X = \sigma_z \otimes \mathbb{I}$ will be measured. Think of $\mathbb{I} \otimes M_2$ as a small computer memory, capable of storing one bit of information. \mathcal{C} is the commutative algebra generated by $Y = \mathbb{I} \otimes \sigma_z$: the actual memory. α is the automorphism defined by $\alpha(A \otimes B) = B \otimes A$. Then $\alpha^*(\rho)(Y) = \rho(\alpha(Y)) = \rho(X)$: the information that was in the electron prior to measurement has now arrived inside \mathcal{C} .

Of course this is not repeatable: the states of the electron and the computer memory are interchanged, so that a second measurement of X will in general yield a different result.

1.3.2 More General Measurement

For automorphic measurement, we require that $\mathbb{P}_{\alpha^*(\sigma), Y} = \mathbb{P}_{\sigma, X}$ holds for all σ in $\mathcal{S}(\mathcal{D})$. But this may only be necessary for a restricted class of states σ in $\mathcal{S}(\mathcal{D})$. An experiment often consists of two parts: a system \mathcal{A} to be examined in an unknown state ρ and a measurement apparatus \mathcal{B} in a known default-state⁸ τ . \mathcal{A} contains the observable X that is to be measured. \mathcal{B} contains some ‘pointer-observable’ Y . Automorphic time evolution on $\mathcal{D} = \mathcal{A} \otimes \mathcal{B}$ may now take place in such a way that $\mathbb{P}_{\alpha^*(\rho \otimes \tau), \mathbb{I} \otimes Y} = \mathbb{P}_{\rho \otimes \tau, X \otimes \mathbb{I}} = \mathbb{P}_{\rho, X}$. Then Y is observed, directly or indirectly. The set of states σ in $\mathcal{S}(\mathcal{D})$ for which $\mathbb{P}_{\alpha^*(\sigma), \mathbb{I} \otimes Y} = \mathbb{P}_{\sigma, X \otimes \mathbb{I}}$ must hold is in this case $\{\sigma = \rho \otimes \tau \mid \rho \in \mathcal{S}(\mathcal{A})\}$.

We define $\mathbf{M}^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ by $\mathbf{M}^*(\rho) \stackrel{\text{def}}{=} \alpha^*(\rho \otimes \tau)$. Because it is affine, \mathbf{M}^* can be extended to a linear mapping $\mathcal{A}^* \rightarrow (\mathcal{A} \otimes \mathcal{B})^*$ on all continuous linear functionals on \mathcal{A} . It is therefore the dual of a linear map $\mathbf{M} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$. (Hence the notation \mathbf{M}^* .) Because \mathbf{M}^* respects normalization, \mathbf{M} is unital: $\mathbf{M}(\mathbb{I}) = \mathbb{I}$. And because \mathbf{M}^* maps

⁸This default-state τ certainly need not be pure.

states to states, \mathbf{M} is positive: $B \geq 0 \Rightarrow \mathbf{M}(B) \geq 0$. \mathbf{M} is even completely positive. (See chapter 3.)

In summary, an affine map $\mathbf{M}^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ of the form $\mathbf{M}^*(\rho) \stackrel{\text{def}}{=} \alpha^*(\rho \otimes \tau)$ is by definition a *perfect* measurement iff $\mathbb{P}_{\mathbf{M}^*(\rho), Y} = \mathbb{P}_{\rho, X}$ for all $\rho \in \mathcal{S}(\mathcal{A})$.

Example

This example is basically due to Hepp (see [Hep]): let $\mathcal{D} = M_2 \otimes M_2$. $M_2 \otimes \mathbb{I}$ is as an electron with spin $X = \sigma_z \otimes \mathbb{I}$. It is in an unknown state $\rho \in \mathcal{S}(M_2)$. $\mathbb{I} \otimes M_2$ is again a computer memory, capable of storing one bit of information. It is in the default-state $\tau = \psi_+$. The actual memory \mathcal{C} is generated by $Y = \mathbb{I} \otimes \sigma_z$.

In the notation of page iv, α is induced by the unitary operator $\mathbf{P}_+ \otimes \mathbb{I} + \mathbf{P}_- \otimes \sigma_x$. It is the controlled not-gate. One easily checks that $\alpha(Y)$ equals $\sigma_z \otimes \sigma_z$ and not X . However, for all ρ in $\mathcal{S}(M_2)$ we have

$$\mathbf{M}^*(\rho)(Y) = \alpha^*(\rho \otimes \tau)(Y) = \rho \otimes \psi_+(\sigma_z \otimes \sigma_z) = \rho \otimes \psi_+(\sigma_z \otimes \mathbb{I}) = \rho(X).$$

\mathcal{C} can now observe Y directly, finding the same probability distribution that direct observation of X would have delivered.

Unbiased Measurement

We will also consider *unbiased* measurement: the average is transferred from X to Y , but not necessarily the entire probability distribution. Automorphic measurement is automatically perfect.

Now $\mathbf{M}^*(\rho)(Y) = \rho(X) \quad \forall \rho \in \mathcal{S}(\mathcal{A}) \Leftrightarrow \mathbf{M}(Y) = X$. This characterizes unbiased measurement. In contrast to automorphic measurement, this does not imply $\mathbf{M}(f(Y)) = f(X)$ for all continuous f . This means that in general $\mathbb{P}_{\mathbf{M}^*(\rho), Y} \neq \mathbb{P}_{\rho, X}$, although the averages do coincide. For perfect measurement, equality holds: $\mathbb{P}_{\mathbf{M}^*(\rho), Y} = \mathbb{P}_{\rho, X}$ or equivalently $\mathbf{M}(f(Y)) = f(X) \quad \forall f \in \mathcal{C}(\mathbf{Spec}(X))$. Yet even for perfect measurement, it may well be that $\alpha(Y) \neq X$. In the example above, \mathbf{M}^* is an unbiased measurement of X : $\mathbf{M}(Y) = X$. It is even perfect: $\mathbf{M}(f(Y)) = f(X)$ for functions on $\mathbf{Spec}(X)$. But it is not automorphic: $\alpha(\mathbb{I} \otimes \sigma_z) = \sigma_z \otimes \sigma_z$, so $\alpha(Y) \neq X$.

In summary: all automorphic measurements are perfect. All perfect measurements are unbiased. And both statements cannot be reversed.

1.4 State Reduction

Before plunging into the generalities of state reduction, let us look at an example.

1.4.1 An Example of Reduction

Let $\mathcal{D} = M_2 \otimes M_2 \otimes M_2$. Again, $M_2 \otimes \mathbb{I} \otimes \mathbb{I}$ is an electron in unknown state $\rho \in \mathcal{S}(M_2)$. It has spin $X = \sigma_z \otimes \mathbb{I} \otimes \mathbb{I}$ to be measured. $\mathbb{I} \otimes M_2 \otimes M_2$ is a computer memory in default state $\tau = \psi_+ \otimes \psi_+$. It is capable of storing two bits of information. \mathcal{C} is the commutative algebra generated by $Y_1 = \mathbb{I} \otimes \sigma_z \otimes \mathbb{I}$ and $Y_2 = \mathbb{I} \otimes \mathbb{I} \otimes \sigma_z$.

First, in exactly the same way as above, X is measured and the information is stored on Y_1 : α_{t_1} is induced by the unitary operator $\mathbf{P}_+ \otimes \mathbb{I} \otimes \mathbb{I} + \mathbf{P}_- \otimes \sigma_x \otimes \mathbb{I}$. Then another, similar measurement of X is performed using Y_2 as pointer: $\alpha_{t_2-t_1}$ is induced⁹ by $\mathbf{P}_+ \otimes \mathbb{I} \otimes \mathbb{I} + \mathbf{P}_- \otimes \mathbb{I} \otimes \sigma_x$.

Finally, \mathcal{C} directly observes Y_1 and Y_2 in state $\mathbf{M}^*(\rho) = \alpha_{t_2}^*(\rho \otimes \tau)$. The results will be distributed according to the probability distribution $\mathbb{P}_{\mathbf{M}^*(\rho), Y_1, Y_2}$:

$$\begin{aligned} \mathbf{M}^*(\rho)(\mathbb{I} \otimes \mathbf{P}_+ \otimes \mathbf{P}_+) &= \rho(\mathbf{P}_+) , \quad 0 &= \mathbf{M}^*(\rho)(\mathbb{I} \otimes \mathbf{P}_+ \otimes \mathbf{P}_-) \\ \mathbf{M}^*(\rho)(\mathbb{I} \otimes \mathbf{P}_- \otimes \mathbf{P}_+) &= 0 , \quad \rho(\mathbf{P}_-) &= \mathbf{M}^*(\rho)(\mathbb{I} \otimes \mathbf{P}_- \otimes \mathbf{P}_-) \end{aligned}$$

In other words, \mathcal{C} observes:

$Y_1 = +1$ and $Y_2 = +1$ with probability $\mathbb{P}_{\rho, X}(\{+1\})$

$Y_1 = -1$ and $Y_2 = -1$ with probability $\mathbb{P}_{\rho, X}(\{-1\})$

$Y_1 = -1$ and $Y_2 = +1$ with probability 0

$Y_1 = +1$ and $Y_2 = -1$ with probability 0.

\mathcal{C} may interpret this correlation causally: the first measurement outcome influences the second. Correlation can also be seen with the help of the reduced state:

$$\begin{aligned} (\mathbf{M}^*(\rho))_{\mathbb{I} \otimes \mathbf{P}_+ \otimes \mathbb{I}}(A) &= \\ &= \frac{\rho \otimes \tau \left(\alpha_{t_2} \left((\mathbb{I} \otimes \mathbf{P}_+ \otimes \mathbb{I}) A (\mathbb{I} \otimes \mathbf{P}_+ \otimes \mathbb{I}) \right) \right)}{\mathbf{M}^*(\rho)(\mathbb{I} \otimes \mathbf{P}_+ \otimes \mathbb{I})} \\ &= \frac{\rho \otimes \tau \left(((\mathbf{P}_+ \otimes \mathbf{P}_+ + \mathbf{P}_- \otimes \mathbf{P}_-) \otimes \mathbb{I}) \alpha_{t_2}(A) ((\mathbf{P}_+ \otimes \mathbf{P}_+ + \mathbf{P}_- \otimes \mathbf{P}_-) \otimes \mathbb{I}) \right)}{\rho \otimes \tau (\mathbf{P}_+ \otimes \mathbf{P}_+ \otimes \mathbb{I} + \mathbf{P}_- \otimes \mathbf{P}_- \otimes \mathbb{I})} \\ &= \frac{\rho \otimes \tau \left(((\mathbf{P}_+ \otimes \mathbb{I}) \otimes \mathbb{I}) \alpha_{t_2}(A) ((\mathbf{P}_+ \otimes \mathbb{I}) \otimes \mathbb{I}) \right)}{\rho(\mathbf{P}_+)} \\ &= \alpha_{t_2}^*(\rho_{\mathbf{P}_+} \otimes \tau)(A) \\ &= \mathbf{M}^*(\rho_{\mathbf{P}_+}) \end{aligned}$$

In the third step, we have made special use of $\tau = \psi_+ \otimes \psi_+$. According to the discussion following definition 1, the above equation has the following significance:

Observations made by \mathcal{C} , conditioned on the first measurement outcome $Y_1 = +1$, will be as if the electron had originally been in the reduced state $\rho_{\mathbf{P}_+}$.

⁹In realistic models, α_{t_1} and $\alpha_{t_2-t_1}$ belong to the same dynamical semi-group, so that $\alpha_{t_1} \circ \alpha_{t_2-t_1} = \alpha_{t_2-t_1} \circ \alpha_{t_1}$. This is satisfied in this example.

Since $\rho_{\mathbf{P}_+} = \psi_+$ for all ρ in $\mathcal{S}(M_2)$, and because $\mathbf{M}^*(\psi_+) = \psi_+ \otimes \psi_+$, this explains that if, according to \mathcal{C} , the first measurement yields $+1$, so does the second.

Nota Bene

One would be tempted to pose the following question:

Suppose that \mathcal{C} observes $Y_1 = +1$ at time t_1 . Does \mathcal{C} then necessarily observe $Y_1 = +1$ at time t_2 ?

This question is metaphysical in nature because it cannot be answered by experiment. At time t_2 , how do you know what you observed at time t_1 ? You must consult some memory.

Any experiment one could possibly devise involves a memory (possibly external; a piece of paper or a hard-disk) storing information on Y_1 . Above, this memory is simply Y_1 itself. And just as above, the result of observing this memory at time t_2 will never yield discrepancies within the memory, independent of the observation made at time t_1 .

1.4.2 Reduction as a Consequence of Measurement

Many examples of the kind above have been described ([Hep], [Böh, p. 292], [B&J, p. 678]). But in fact, reduction is not just *possible*, as has been known for a long time (see [Neu]). It is a *necessary* consequence of transferring information from X to some pointer Y .

The Origin of Reduction and Collapse

Let $\mathbf{M} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$ be such that $\mathbf{M}^*(\rho) = \alpha(\rho \otimes \tau)$ for some automorphism α and $\tau \in \mathcal{S}(\mathcal{B})$. Suppose \mathbf{M} is a perfect measurement of $X \in \mathcal{A}$ with¹⁰ pointer $Y \in \mathcal{A} \otimes \mathcal{B}$. Suppose that \mathcal{A} and \mathcal{B} are von Neumann algebras, as in the previous example. Then X and Y have projection valued measures $V \mapsto \mathbf{P}(V)$ and $W \mapsto \mathbf{Q}(W)$ respectively. If ρ is normal, then $\mathbb{P}_{\mathbf{M}^*(\rho), Y} = \mathbb{P}_{\rho, X}$ implies $\mathbf{M}^*(\rho)(\mathbf{Q}(V)) = \rho(\mathbf{P}(V))$ for all subsets V of $\text{Spec}(X)$: the spectral projections of X are measured, using the corresponding ones of Y as pointers. In this situation we can apply:

Proposition 3 (Reduction) *Let \mathcal{A} and \mathcal{B} be C^* -algebras, $\mathbf{P} \in \mathcal{A}$ and $\mathbf{Q} \in \mathcal{A} \otimes \mathcal{B}$ projections. For $\rho \in \mathcal{S}(\mathcal{A})$, let $\mathbf{M}^*(\rho) \stackrel{\text{def}}{=} \alpha^*(\rho \otimes \tau)$ for some automorphism α of $\mathcal{A} \otimes \mathcal{B}$ and $\tau \in \mathcal{S}(\mathcal{B})$. Suppose $\mathbf{M}^*(\rho)(\mathbf{Q}) = \rho(\mathbf{P})$ for all $\rho \in \mathcal{S}(\mathcal{A})$. Then for any $\rho \in \mathcal{S}(\mathcal{A})$:*

$$(\mathbf{M}^*(\rho))_{\mathbf{Q}} = \mathbf{M}^*(\rho_{\mathbf{P}}).$$

Proof:

By moving to the GNS-representation, we may assume $\rho \otimes \tau$ to be a vector state $|\psi\rangle$. Now by assumption,

$$\alpha^*(\rho_{\mathbf{P}} \otimes \tau)(\mathbf{Q}) = \mathbf{M}^*(\rho_{\mathbf{P}})(\mathbf{Q}) = \rho_{\mathbf{P}}(\mathbf{P}) = 1.$$

¹⁰This includes automorphic measurement if \mathcal{B} happens to be \mathbb{I} and \mathbf{M} automorphic.

Since $(\rho_{\mathbf{P}} \otimes \tau) = (\rho \otimes \tau)_{\mathbf{P} \otimes \mathbb{I}}$ corresponds to the vector state $\frac{\mathbf{P} \otimes \mathbb{I}|\psi\rangle}{\|\mathbf{P} \otimes \mathbb{I}|\psi\rangle\|}$, we have

$$\alpha^*(\rho_{\mathbf{P}} \otimes \tau)(\mathbf{Q}) = \frac{\langle \mathbf{P} \otimes \mathbb{I} \psi | \alpha(\mathbf{Q}) | \mathbf{P} \otimes \mathbb{I} \psi \rangle}{\|\mathbf{P} \otimes \mathbb{I} \psi\|^2} = 1. \quad (1.2)$$

$\alpha(\mathbf{Q})$ is a projection. Therefore equation (1.2) implies

$$\|\alpha(\mathbf{Q})\mathbf{P} \otimes \mathbb{I}|\psi\rangle\|^2 = \|\mathbf{P} \otimes \mathbb{I}|\psi\rangle\|^2$$

which entails, again because $\alpha(\mathbf{Q})$ is a projection, that

$$\alpha(\mathbf{Q})\mathbf{P} \otimes \mathbb{I}|\psi\rangle = \mathbf{P} \otimes \mathbb{I}|\psi\rangle. \quad (1.3)$$

In a similar fashion, $\mathbf{M}^*(\rho_{\mathbb{I}-\mathbf{P}})(\mathbf{Q}) = \rho_{\mathbb{I}-\mathbf{P}}(\mathbf{P}) = 0$ leads to

$$\alpha(\mathbf{Q})((\mathbb{I} - \mathbf{P}) \otimes \mathbb{I})|\psi\rangle = 0. \quad (1.4)$$

Equations 1.3 and 1.4 imply

$$\alpha(\mathbf{Q})|\psi\rangle = \mathbf{P} \otimes \mathbb{I}|\psi\rangle.$$

Thus for all $D \in \mathcal{A} \otimes \mathcal{B}$:

$$\begin{aligned} (\mathbf{M}^*(\rho))_{\mathbf{Q}}(D) &= \frac{\rho \otimes \tau(\alpha(\mathbf{Q}D\mathbf{Q}))}{\mathbf{M}^*(\rho)(\mathbf{Q})} \\ &= \frac{\langle \alpha(\mathbf{Q})\psi | \alpha(D) | \alpha(\mathbf{Q})\psi \rangle}{\rho(\mathbf{P})} \\ &= \frac{\langle (\mathbf{P} \otimes \mathbb{I})\psi | \alpha(D) | (\mathbf{P} \otimes \mathbb{I})\psi \rangle}{\rho \otimes \tau(\mathbf{P} \otimes \mathbb{I})} \\ &= (\rho \otimes \tau)_{\mathbf{P} \otimes \mathbb{I}}(\alpha(D)) = \alpha^*(\rho_{\mathbf{P}} \otimes \tau)(D) \\ &= \mathbf{M}^*(\rho_{\mathbf{P}})(D). \end{aligned}$$

q.e.d.

State Reduction

This has two major consequences. The first is subjective:

Suppose that a perfect measurement of $X \in \mathcal{D}$ is performed with pointer $Y \in \mathcal{C} \subset \mathcal{D}$. If \mathcal{D} was in state $\rho \in \mathcal{S}(\mathcal{A})$ before measurement, then all observations made by \mathcal{C} after measurement, conditioned on the observation that the measurement outcome Y is in the set V , will be as if the system had originally been in the reduced state $\rho_{\mathbf{P}(V)}$ instead of ρ .

We now have an interpretation of the reduced state $\rho_{\mathbf{P}(V)}$ outside Y' . Perhaps the following commutative diagram says more than a thousand words:

$$\begin{array}{ccccc}
\rho & \longrightarrow & \mathbf{M}^*(\rho) & \longrightarrow & \mathbb{P}_{\mathbf{M}^*(\rho), Y} \\
\downarrow \text{reduction} & & \downarrow & & \downarrow \text{conditioning} \\
\rho_{\mathbf{P}(V)} & \longrightarrow & (\mathbf{M}^*(\rho))_{\mathbf{Q}(V)} & \longrightarrow & \mathbb{P}_{\mathbf{M}^*(\rho), Y}(\bullet \mid [Y \text{ in } V]) \\
& & \text{=} & & \text{=} \\
& & \mathbf{M}^*(\rho_{\mathbf{P}(V)}) & \longrightarrow & \mathbb{P}_{\mathbf{M}^*(\rho_{\mathbf{P}(V)}), Y}
\end{array}$$

The map $\rho \rightarrow \rho_{\mathbf{P}(V)}$ is called state reduction. The fact that we observe state reduction (left hand side of the diagram) results from harmless conditioning on the physically relevant probability distributions (right hand side).

State Collapse

The second consequence is objective. It is summarized in the diagram below:

$$\begin{array}{ccc}
\rho & \longrightarrow & \mathbf{M}^*(\rho) \\
\downarrow \text{collapse} & & \text{=} \text{ on } Y' \\
\mathbf{C}^*(\rho) & \longrightarrow & \mathbf{M}^* \circ \mathbf{C}^*(\rho)
\end{array}$$

Corollary 3.1 (Collapse) *Let \mathcal{A} and \mathcal{B} be von Neumann algebras, $X \in \mathcal{A}$, $Y \in \mathcal{A} \otimes \mathcal{B}$ Hermitean. Let $\{V_i \mid i \in I\}$ be any countable decomposition of $\mathbf{Spec}(X)$. Let $\mathbf{M}^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ be defined by $\mathbf{M}^*(\rho) \stackrel{\text{def}}{=} \alpha^*(\rho \otimes \tau)$ for some automorphism α of $\mathcal{A} \otimes \mathcal{B}$ and normal state $\tau \in \mathcal{S}(\mathcal{B})$. Suppose \mathbf{M}^* is a perfect measurement of X with pointer Y , i.e. $\mathbb{P}_{\mathbf{M}^*(\rho), Y} = \mathbb{P}_{\rho, X}$ for all $\rho \in \mathcal{S}(\mathcal{A})$. Then for all normal $\rho \in \mathcal{S}(\mathcal{A})$, and for all $D \in Y'$:*

$$\mathbf{M}^*(\rho)(D) = (\mathbf{M}^* \circ \mathbf{C}^*)(\rho)(D),$$

where \mathbf{C}^* is the collapse operation for X and $\{V_i \mid i \in I\}$, as in definition 2.

Proof:

Let $V \mapsto \mathbf{P}(V)$ and $V \mapsto \mathbf{Q}(V)$ be the spectral measures of X and Y respectively. Since $\bigcup_I V_i = \mathbf{Spec}(X)$, we have $\sum_I \mathbf{Q}(V_i) = \mathbb{I}$. Suppose $[D, Y] = 0$. Then also $[D, \mathbf{Q}(V_i)] = 0$ for all $i \in I$. Therefore

$$\begin{aligned}
\mathbf{M}^*(\rho)(D) &= \mathbf{M}^*(\rho)\left(\left(\sum_I \mathbf{Q}(V_i)\right)D\right) \\
&= \mathbf{M}^*(\rho)\left(\sum_I \mathbf{Q}(V_i)D\mathbf{Q}(V_i)\right) \\
&= \sum_I \mathbb{P}_{\mathbf{M}^*(\rho), Y}(V_i) \cdot (\mathbf{M}^*(\rho))_{\mathbf{Q}(V_i)}(D)
\end{aligned}$$

$$\begin{aligned}
&= \sum_I \mathbb{P}_{\rho, X}(V_i) \cdot \mathbf{M}^*(\rho_{\mathbf{P}(V_i)})(D) \\
&= \sum_I \rho(\mathbf{P}(V_i) \mathbf{M}(D) \mathbf{P}(V_i)) \\
&= (\mathbf{M}^* \circ \mathbf{C}^*)(\rho)(D).
\end{aligned}$$

q.e.d.

Normally, a system \mathcal{A} will be examined by an observer $\mathcal{C} \subset \mathcal{B}$ outside \mathcal{A} . This means that Y is of the form $\mathbb{I} \otimes \tilde{Y}$. Then all $A \in \mathcal{A} \otimes \mathcal{B}$ of the form $\tilde{A} \otimes \mathbb{I}$ will commute with the pointer. So regarded as a state on the examined system $\mathcal{A} \otimes \mathbb{I}$, we have

$$\mathbf{M}^*(\rho) = \mathbf{M}^* \circ \mathbf{C}^*(\rho).$$

In other words:

Suppose that a perfect measurement of $X \in \mathcal{A}$ is performed by an observer $\mathcal{C} \subset \mathcal{B}$ outside \mathcal{A} , using a pointer $Y \in \mathbb{I} \otimes \mathcal{B}$. Then all measurements of any $A \in \mathcal{A}$ made by any second observer $\tilde{\mathcal{C}}$ will be as if the system had originally been in the collapsed state $\mathbf{C}^(\rho)$ instead of ρ .*

For example, suppose that $\mathcal{A} = M_2$ is in vector state $\alpha|\psi_+\rangle + \beta|\psi_-\rangle$, and σ_z is measured perfectly by an outside observer. Then all subsequent measurement of \mathcal{A} will be as if \mathcal{A} had originally been in the mixed state $|\alpha|^2 \cdot |\psi_+\rangle\langle\psi_+| + |\beta|^2 \cdot |\psi_-\rangle\langle\psi_-|$.

Summary

Reduction is subjective. It involves only one observer. Reduction occurs after both direct and indirect observation.

Collapse is objective. It involves at least two observers. Note the crucial role of $[A, Y] = 0$: if the first observer had been inside \mathcal{A} instead of outside, none of the above would hold. Collapse only occurs with indirect observation. Both are not just possible, but necessary consequences of measurement.

1.4.3 Imperfect Reduction after Imperfect Measurement

Suppose an unbiased measurement is not perfect, but still rather good. Then one does not expect a perfect reduction, but still a rather good one. Proposition (3) allows such a generalized version. In contrast to generalized collapse (which comes along quite naturally), generalized reduction is rather thorny and uncomfortable. But in the end, if we work hard enough, we do obtain a hard estimate of the reduction, even for biased measurement:

Proposition 4 (Generalized Reduction) *Let \mathcal{A} and \mathcal{B} be C^* -algebras, $\mathbf{P} \in \mathcal{A}$ and $\mathbf{Q} \in \mathcal{A} \otimes \mathcal{B}$ projections. For $\rho \in \mathcal{S}(\mathcal{A})$, let $\mathbf{M}^*(\rho) \stackrel{\text{def}}{=} \alpha^*(\rho \otimes \tau)$ for some automorphism α of $\mathcal{A} \otimes \mathcal{B}$ and $\tau \in \mathcal{S}(\mathcal{B})$. Suppose there is a $\Delta \geq 0$ such that $|\mathbf{M}^*(\rho)(\mathbf{Q}) - \rho(\mathbf{P})| \leq \Delta$ for all $\rho \in \mathcal{S}(\mathcal{A})$. Then for any $\rho \in \mathcal{S}(\mathcal{A})$:*

$$\|(\mathbf{M}^*(\rho))_{\mathbf{Q}} - \mathbf{M}^*(\rho_{\mathbf{P}})\| \leq \frac{\sqrt{\Delta}}{\mathbf{M}^*(\rho)(\mathbf{Q})} \left(1 + 2\sqrt{\Delta} + \sqrt{1 + (1 + 2\sqrt{\Delta})^2} \right)$$

Proof:

For notational convenience, define $\mathbf{P}_0 = \mathbf{P}$, $\mathbf{P}_1 = \mathbb{I} - \mathbf{P}$. Define $\epsilon_i^2 = |\rho_{\mathbf{P}_i}(\mathbf{P}) - \mathbf{M}^*(\rho_{\mathbf{P}_i})(\mathbf{Q})| = |\delta_{i,0} - \mathbf{M}^*(\rho_{\mathbf{P}_i})(\mathbf{Q})|$. Here ϵ_i^2 is the probability that a measurement of \mathbf{P} in state $\rho_{\mathbf{P}_i}$ yields the wrong outcome. The ϵ_i depend on ρ , but are always less than $\sqrt{\Delta}$. By constructing the GNS representation of $\rho \otimes \tau$, we may assume $\rho \otimes \tau$ to be a vector state $|\psi\rangle$. Then the ϵ_i have geometrical significance: they regulate the length of the vector

$$\begin{aligned} |\chi_i\rangle &\stackrel{\text{def}}{=} \alpha(\mathbf{Q})\mathbf{P}_i \otimes \mathbb{I}|\psi\rangle - \mathbf{P}\mathbf{P}_i \otimes \mathbb{I}|\psi\rangle \\ &= (\alpha(\mathbf{Q}) - \delta_{i,0})\mathbf{P}_i \otimes \mathbb{I}|\psi\rangle \end{aligned} \quad (1.5)$$

by

$$\begin{aligned} \|\chi_i\|^2 &\stackrel{\text{def}}{=} \|\alpha(\mathbf{Q})\mathbf{P}_i \otimes \mathbb{I}|\psi\rangle - \delta_{i,0}\mathbf{P}_i \otimes \mathbb{I}|\psi\rangle\|^2 \\ &= \langle \mathbf{P}_i \otimes \mathbb{I} | (\alpha(\mathbf{Q}) - \delta_{i,0})^\dagger (\alpha(\mathbf{Q}) - \delta_{i,0}) | \mathbf{P}_i \otimes \mathbb{I} \rangle \\ &= \langle \mathbf{P}_i \otimes \mathbb{I} | (1 - 2\delta_{i,0})\alpha(\mathbf{Q}) + \delta_{i,0} | \mathbf{P}_i \otimes \mathbb{I} \rangle \\ &= (1 - 2\delta_{i,0})\rho \otimes \tau((\mathbf{P}_i \otimes \mathbb{I})\alpha(\mathbf{Q})(\mathbf{P}_i \otimes \mathbb{I})) + \delta_{i,0}\rho \otimes \tau(\mathbf{P}_i \otimes \mathbb{I}) \\ &= \rho(\mathbf{P}_i)((1 - 2\delta_{i,0})\alpha^*(\rho \otimes \tau_{\mathbf{P}_i \otimes \mathbb{I}})(\mathbf{Q}) + \delta_{i,0}) \\ &= \rho(\mathbf{P}_i)(|\alpha^*(\rho \otimes \tau_{\mathbf{P}_i \otimes \mathbb{I}})(\mathbf{Q}) - \delta_{i,0}|) \\ &= \rho(\mathbf{P}_i)(|\mathbf{M}^*(\rho_{\mathbf{P}_i})(\mathbf{Q}) - \delta_{i,0}|) \\ &= \epsilon_i^2 \rho(\mathbf{P}_i). \end{aligned} \quad (1.6)$$

With this we will estimate

$$\begin{aligned} \mathbf{M}^*(\rho)(\mathbf{Q}X\mathbf{Q}) - \mathbf{M}^*(\rho_{\mathbf{P}})(X) \cdot \mathbf{M}^*(\rho)(\mathbf{Q}) &= \\ &= \alpha^*(\rho \otimes \tau)(\mathbf{Q}X\mathbf{Q}) - \alpha^*(\rho_{\mathbf{P}} \otimes \tau)(X) \cdot \alpha^*(\rho \otimes \tau)(\mathbf{Q}) \\ &= \alpha^*(\rho \otimes \tau)(\mathbf{Q}(X - \alpha^*(\rho_{\mathbf{P}} \otimes \tau)(X))\mathbf{Q}) \\ &= \rho \otimes \tau(\alpha(\mathbf{Q})\alpha(X - \alpha^*(\rho_{\mathbf{P}} \otimes \tau)(X))\alpha(\mathbf{Q})) \\ &= \sum_{k,l} \rho \otimes \tau(\mathbf{P}_k \otimes \mathbb{I}\alpha(\mathbf{Q})\alpha(X - \alpha^*(\rho_{\mathbf{P}} \otimes \tau)(X))\alpha(\mathbf{Q})\mathbf{P}_l \otimes \mathbb{I}) \\ &= \sum_{k,l} \langle \alpha(\mathbf{Q})\mathbf{P}_k \otimes \mathbb{I} | \psi \rangle \langle \alpha(X - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X)) | \alpha(\mathbf{Q})\mathbf{P}_l \otimes \mathbb{I} | \psi \rangle \\ &= \sum_{k,l} \langle \delta_{k,0}\mathbf{P}_k \otimes \mathbb{I} | \psi \rangle + \chi_k | \alpha(X - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X)) | \delta_{l,0}\mathbf{P}_l \otimes \mathbb{I} | \psi \rangle + \chi_l. \end{aligned} \quad (1.7)$$

The last step goes by definition of χ_k : from equation (1.5), we see that $\alpha(\mathbf{Q})\mathbf{P}_k \otimes \mathbb{I} | \psi \rangle = \delta_{k,0}\mathbf{P}_k \otimes \mathbb{I} | \psi \rangle + \chi_k$. We will examine the smallness of each term separately.

$$\begin{aligned} \langle \mathbf{P} \otimes \mathbb{I} | \psi \rangle \langle \alpha(X - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X)) | \mathbf{P} \otimes \mathbb{I} | \psi \rangle &= \\ &= \rho \otimes \tau(\mathbf{P} \otimes \mathbb{I}\alpha(X)\mathbf{P} \otimes \mathbb{I}) - \rho \otimes \tau(\mathbf{P} \otimes \mathbb{I})\alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X) \\ &= 0. \end{aligned} \quad (1.8)$$

That's one down. We will estimate the cross-terms with the Cauchy-Schwarz inequality. For that, we need the length of both $\|\chi_k\|$ and the vector

$$\begin{aligned}
& \|\alpha(X - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X))\mathbf{P} \otimes \mathbb{I}\psi\|^2 = \\
&= \langle \mathbf{P} \otimes \mathbb{I}\psi | \alpha(X - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X))^\dagger \\
&\quad \alpha(X - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X)) | \mathbf{P} \otimes \mathbb{I}\psi \rangle \\
&= \langle \mathbf{P} \otimes \mathbb{I}\psi | \alpha(X)^\dagger \alpha(X) - 2\Re(\alpha(X)^\dagger \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X)) + \\
&\quad + |\alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X)|^2 | \mathbf{P} \otimes \mathbb{I}\psi \rangle \\
&= \langle \mathbf{P} \otimes \mathbb{I}\psi | \alpha(X)^\dagger \alpha(X) | \mathbf{P} \otimes \mathbb{I}\psi \rangle - \frac{|\langle \mathbf{P} \otimes \mathbb{I}\psi | \alpha(X) | \mathbf{P} \otimes \mathbb{I}\psi \rangle|^2}{\|\mathbf{P} \otimes \mathbb{I}\psi\|^2} \\
&= \langle \mathbf{P} \otimes \mathbb{I}\psi | \alpha(X)^\dagger (\mathbb{I} - \frac{\langle \mathbf{P} \otimes \mathbb{I}\psi | \alpha(X) | \mathbf{P} \otimes \mathbb{I}\psi \rangle}{\langle \mathbf{P} \otimes \mathbb{I}\psi | \mathbf{P} \otimes \mathbb{I}\psi \rangle}) \alpha(X) | \mathbf{P} \otimes \mathbb{I}\psi \rangle \\
&\leq \|\mathbf{P} \otimes \mathbb{I}\psi\|^2 \|X\|^2.
\end{aligned} \tag{1.9}$$

Now that we have the length of both vectors, we see by Cauchy-Schwarz:

$$\|\langle \chi_k | \alpha(X - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X))\mathbf{P} \otimes \mathbb{I}\psi \rangle\| \leq \|X\| \cdot \|\mathbf{P} \otimes \mathbb{I}\psi\| \epsilon_k \|\mathbf{P}_k \otimes \mathbb{I}\psi\|. \tag{1.10}$$

And similarly

$$\|\langle \mathbf{P} \otimes \mathbb{I}\psi | \alpha(X - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X)) | \chi_l \rangle\| \leq \|X\| \cdot \|\mathbf{P} \otimes \mathbb{I}\psi\| \epsilon_l \|\mathbf{P}_l \otimes \mathbb{I}\psi\|. \tag{1.11}$$

Finally, from $\|\alpha(X) - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X)\| \leq 2\|X\|$ we see that

$$\|\langle \chi_k | \alpha(X - \alpha^*(\rho \otimes \tau_{\mathbf{P} \otimes \mathbb{I}})(X)) | \chi_l \rangle\| \leq 2\|X\| \epsilon_k \|\mathbf{P}_k \otimes \mathbb{I}\psi\| \epsilon_l \|\mathbf{P}_l \otimes \mathbb{I}\psi\|. \tag{1.12}$$

Putting inequalities (1.8), (1.10), (1.11) and (1.12) into (1.7), we finally obtain

$$\begin{aligned}
& \|\mathbf{M}^*(\rho)(\mathbf{Q}X\mathbf{Q}) - \mathbf{M}^*(\rho_{\mathbf{P}})(X) \cdot \mathbf{M}^*(\rho)(\mathbf{Q})\| \leq \\
&\leq 2\|X\| (\epsilon_0 \|\mathbf{P}_0 \otimes \mathbb{I}\psi\| + \epsilon_1 \|\mathbf{P}_1 \otimes \mathbb{I}\psi\|) + \\
&\quad 2\|X\| (\epsilon_0 \|\mathbf{P}_0 \otimes \mathbb{I}\psi\| + \epsilon_1 \|\mathbf{P}_1 \otimes \mathbb{I}\psi\|)^2 \\
&= 2\|X\| (\epsilon_0 \|\mathbf{P}_0 \otimes \mathbb{I}\psi\| + \epsilon_1 \|\mathbf{P}_1 \otimes \mathbb{I}\psi\|) \times \\
&\quad (\|\mathbf{P}_0 \otimes \mathbb{I}\psi\| + \epsilon_0 \|\mathbf{P}_0 \otimes \mathbb{I}\psi\| + \epsilon_1 \|\mathbf{P}_1 \otimes \mathbb{I}\psi\|).
\end{aligned}$$

To estimate this last expression, note that \mathbf{P}_0 and \mathbf{P}_1 are complementary projections, and $|\psi\rangle$ is of norm one. Therefore, there exists an angle θ such that $\cos \theta = \|\mathbf{P}_0 \otimes \mathbb{I}\psi\|$ and $\sin \theta = \|\mathbf{P}_1 \otimes \mathbb{I}\psi\|$. Since both $\epsilon_0, \epsilon_1 \leq \sqrt{\Delta}$, we have

$$\left\| \frac{\mathbf{M}^*(\rho)(\mathbf{Q}X\mathbf{Q})}{\mathbf{M}^*(\rho)(\mathbf{Q})} - \mathbf{M}^*(\rho_{\mathbf{P}})(X) \right\| \leq \frac{\|X\| \sqrt{\Delta}}{\mathbf{M}^*(\rho)(\mathbf{Q})} f(\theta) \tag{1.13}$$

with $f(\theta) = 2(\cos \theta + \sin \theta)(\cos \theta + \sqrt{\Delta}(\cos \theta + \sin \theta))$. With standard analysis and goniometry, one can verify that f takes maximal value $1 + 2\sqrt{\Delta} + \sqrt{1 + (1 + 2\sqrt{\Delta})^2}$, proving the proposition.

q. e. d.

Proposition (3) is contained in the above as the special case $\Delta = 0$. Note that the bound disappears if the probability of observing measurement outcome $+1$ becomes less than $1/2\sqrt{\Delta}$. This means that excellent measurement ($\Delta \sim 0$) without reduction (upon finding $+1$) is not excluded, provided that the probability of outcome $+1$ remains small. Of course there is conservation of misery: the probability of outcome 0 is large, and upon finding 0 there is very good reduction.

Nonetheless, this principle can be used nicely in so-called ‘knowingly reversible measurement’ (see [DAr]). This is a non-perfect measurement, leaving the state fixed with a certain probability. The observer obtains not only a measurement outcome, but also the information whether or not the state is conserved successfully.

1.5 State Collapse

On page 15, we have obtained collapse from reduction in order to show the link between the two. But there is an easier way of proving the necessity of collapse, more suitable for generalization.

1.5.1 Perfect Collapse after Perfect Measurement

The setting is slightly different: an observer $\mathcal{C} \subset \mathcal{B}$ attempts to decide whether a system \mathcal{A} is in state ψ_1 or in ψ_2 . In order to do that, a measurement $\mathbf{M}^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ is performed (of the form $\mathbf{M}^*(\rho) = \alpha^*(\rho \otimes \tau)$) in such a way that observation of a certain pointer-observable $Y \in \mathcal{C}$ yields with certainty y_1 in state $\mathbf{M}^*(\psi_1)$ and y_2 in state $\mathbf{M}^*(\psi_2)$.

Lemma 5 *Let $|\phi_i\rangle$ ($i = 1, 2$) be vector states on some algebra \mathcal{D} . Let $Y \in \mathcal{D}$ be Hermitean such that*

$$\mathbf{Var}_{\phi_i}(Y) = 0 \quad \text{and} \quad \langle \phi_i | Y | \phi_i \rangle = y_i \quad (i = 1, 2)$$

with $y_1 \neq y_2$. Then for all $A \in \mathcal{D}$ such that $[A, Y] = 0$:

$$\langle \phi_1 | A | \phi_2 \rangle = 0.$$

Proof:

$|\phi_1\rangle$ and $|\phi_2\rangle$ must be eigenvectors of Y with eigenvalues y_1 and y_2 . Since $[A, Y] = 0$, A respects the eigenspaces of Y . We therefore have $|\phi_1\rangle \perp |A\phi_2\rangle$:

$$(y_1 - y_2)\langle \phi_1 | A | \phi_2 \rangle = \langle y_1 \phi_1 | A | \phi_2 \rangle - \langle \phi_1 | A | y_2 \phi_2 \rangle = \langle \phi_1 | [Y, A] | \phi_2 \rangle = 0.$$

q. e. d.

This standard result can be used in the following way:

Proposition 6 (Collapse) *Let $\mathbf{M}^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ be of the form $\mathbf{M}^*(\rho) = \alpha^*(\rho \otimes \tau)$ for some automorphism α of $\mathcal{A} \otimes \mathcal{B}$ and $\tau \in \mathcal{S}(\mathcal{B})$. Let ψ_i , ($i = 1, 2$) be vector states on \mathcal{A} , let $|\alpha|^2 + |\beta|^2 = 1$ and let $Y \in \mathcal{A} \otimes \mathcal{B}$ be Hermitean such that*

$$\mathbf{Var}_{\mathbf{M}^*(\psi_i)}(Y) = 0 \quad \text{and} \quad \mathbf{M}^*(\psi_i)(Y) = y_i \quad (i = 1, 2)$$

with $y_1 \neq y_2$. Then for all $A \in \mathcal{A} \otimes \mathcal{B}$ such that $[A, Y] = 0$:

$$\mathbf{M}^*(|\alpha\psi_1 + \beta\psi_2\rangle\langle\alpha\psi_1 + \beta\psi_2|)(A) = \mathbf{M}^*(|\alpha|^2|\psi_1\rangle\langle\psi_1| + |\beta|^2|\psi_2\rangle\langle\psi_2|)(A).$$

Proof:

By the GNS-representation, we assume τ to be a vector state $|\tau\rangle$. Thus

$$\begin{aligned} & |\mathbf{M}^*(|\alpha\psi_1 + \beta\psi_2\rangle\langle\alpha\psi_1 + \beta\psi_2|)(A) - \mathbf{M}^*(|\alpha|^2|\psi_1\rangle\langle\psi_1| + |\beta|^2|\psi_2\rangle\langle\psi_2|)(A)| \\ &= | \langle (\alpha\psi_1 + \beta\psi_2) \otimes \tau | \alpha(A) | (\alpha\psi_1 + \beta\psi_2) \otimes \tau \rangle - \\ &\quad (|\alpha|^2 \langle \psi_1 \otimes \tau | \alpha(A) | \psi_1 \otimes \tau \rangle + |\beta|^2 \langle \psi_2 \otimes \tau | \alpha(A) | \psi_2 \otimes \tau \rangle) | \\ &\leq 2|\alpha||\beta| | \langle \psi_1 \otimes \tau | \alpha(A) | \psi_2 \otimes \tau \rangle | \\ &\leq | \langle \psi_1 \otimes \tau | \alpha(A) | \psi_2 \otimes \tau \rangle |. \end{aligned} \tag{1.14}$$

The last step uses that $2|\alpha||\beta| \leq 1$ since $|\alpha|^2 + |\beta|^2 = 1$.

Finally, we come to lemma (5), here with the vectors $|\psi_i \otimes \tau\rangle$, ($i = 1, 2$) and with the observable $\alpha(Y)$:

$$\psi_i \otimes \tau(\alpha(Y)) = \mathbf{M}^*(\psi_i)(Y) = y_i \quad (1.15)$$

and

$$\begin{aligned} \mathbf{Var}_{\psi_i \otimes \tau}(\alpha(Y)) &= \psi_i \otimes \tau(\alpha(Y)^2) - \psi_i \otimes \tau(\alpha(Y))^2 \\ &= \psi_i \otimes \tau(\alpha(Y^2)) - \psi_i \otimes \tau(\alpha(Y))^2 \\ &= \mathbf{Var}_{\mathbf{M}^*(\psi_i)}(Y) \\ &= 0. \end{aligned} \quad (1.16)$$

Thus $\langle \psi_1 \otimes \tau | \alpha(A) | \psi_2 \otimes \tau \rangle = 0$.

q.e.d.

If Y is of the form $\mathbb{I} \otimes \tilde{Y}$, then $[A \otimes \mathbb{I}, \mathbb{I} \otimes \tilde{Y}] = 0$ for any $A \in \mathcal{A}$. This means that after the measurement is performed on \mathcal{A} in vector state $|\alpha\psi_1 + \beta\psi_2\rangle$, further measurement of any $A \in \mathcal{A}$ by other observers will be as if the state had originally been $|\alpha|^2|\psi_1\rangle\langle\psi_1| + |\beta|^2|\psi_2\rangle\langle\psi_2|$.

Of course the same holds for other observables commuting with Y , such as observables in \mathcal{C} for example, or¹¹ in $\mathcal{A} \otimes \mathcal{C}$. All of this has an immediate generalization for the case of a less perfect measurement, and for observables not quite commuting with Y .

1.5.2 Imperfect Collapse after Imperfect Measurement

Lemma 7 *Let ϕ_i , ($i = 1, 2$) be vector states on some algebra \mathcal{D} . Let Y be a Hermitean element of \mathcal{A} such that $\phi_1(Y) \neq \phi_2(Y)$. Let*

$$\phi_i(Y) = y_i \quad \text{and} \quad \mathbf{Var}_{\phi_i}(Y) = \sigma_i^2 \quad (i = 1, 2)$$

be the expectation and variance of Y in the state ϕ_i , ($i = 1, 2$). If A is a Hermitean observable such that $\|[A, Y]\| \leq \delta\|A\|$, then

$$|\langle \phi_1 | A | \phi_2 \rangle| \leq \frac{\delta + \sigma_1 + \sigma_2}{|y_1 - y_2|} \|A\|.$$

¹¹If the measuring device happens to be classical (\mathcal{B} is commutative), then a complete and rigorous collapse has been achieved. This was proposed by Jauch (see [Jau, p. 174]). Although this is an extremely useful remark (see e.g. [Hep]), I do not hold this to be a *fundamental* solution to the problem of measurement for the following reasons:

- Measurement apparatuses consist of particles. Particles have momentum and position. These do not commute, so Abelian \mathcal{B} can only be an idealization.
- Automorphic time evolution always conserves purity of states (see [Hep, lemma 2]). Even on Abelian algebras.

In chapter 2, we will examine this idealization more closely.

Proof:

For $i = 1, 2$,

$$\sigma_i^2 = \langle \phi_i | Y^\dagger Y | \phi_i \rangle - \langle \phi_i | Y^\dagger | \phi_i \rangle \langle \phi_i | Y | \phi_i \rangle.$$

So

$$\sigma_i^2 = \langle Y \phi_i | \mathbb{I} - \mathbf{P}_{//} | Y \phi_i \rangle$$

where $\mathbf{P}_{//}$ denotes the projection onto the one-dimensional vector-space spanned by $|\phi\rangle$. So $\mathbb{I} - \mathbf{P}_{//}$ is the projection orthogonal to $|\phi_i\rangle$. We denote it by \mathbf{P}_\perp .

From $\mathbf{P}_\perp^2 = \mathbf{P}_\perp^\dagger = \mathbf{P}_\perp$ we see that

$$\sigma_i = \|\mathbf{P}_\perp Y \phi_i\|.$$

Decomposing $Y|\phi_i\rangle$ into components parallel and perpendicular to $|\phi_i\rangle$ we find

$$Y|\phi_i\rangle = \mathbf{P}_{//} Y|\phi_i\rangle + \mathbf{P}_\perp Y|\phi_i\rangle.$$

Denoting $\mathbf{P}_\perp Y|\phi_i\rangle$ by $|\chi_i\rangle$, bearing in mind $\|\chi_i\| = \sigma_i$:

$$Y|\phi_i\rangle = \langle \phi_i | Y^\dagger | \phi_i \rangle |\phi_i\rangle + |\chi_i\rangle = y_i |\phi_i\rangle + |\chi_i\rangle.$$

We use this in the following:

$$\begin{aligned} (y_2 - y_1) \langle \phi_1 | A | \phi_2 \rangle &= \langle \phi_1 | A | y_2 \phi_2 \rangle - \langle y_1 \phi_1 | A | \phi_2 \rangle \\ &= \langle \phi_1 | A Y | \phi_2 \rangle - \langle \phi_1 | A | \chi_2 \rangle - \langle \phi_1 | Y A | \phi_2 \rangle + \langle \chi_1 | A | \phi_2 \rangle \\ &= \langle \phi_1 | [A, Y] | \phi_2 \rangle + \langle \chi_1 | A | \phi_2 \rangle - \langle \phi_1 | A | \chi_2 \rangle. \end{aligned}$$

Estimating with the Cauchy-Schwarz-inequality and the operator norm in each term we obtain

$$|(y_2 - y_1)| \cdot |\langle \phi_1 | A | \phi_2 \rangle| \leq (\delta + \sigma_1 + \sigma_2) \|A\|.$$

q.e.d.

We use lemma (7) in the same way as lemma (5):

Proposition 8 (Generalized Collapse) *Let $\mathbf{M}^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ be of the form $\mathbf{M}^*(\rho) = \alpha^*(\rho \otimes \tau)$ for some automorphism α of $\mathcal{A} \otimes \mathcal{B}$ and $\tau \in \mathcal{S}(\mathcal{B})$. Let ψ_i , ($i = 1, 2$) be vector states on \mathcal{A} , let $|\alpha|^2 + |\beta|^2 = 1$ and let $Y \in \mathcal{A} \otimes \mathcal{B}$ be Hermitean such that*

$$\mathbf{Var}_{\mathbf{M}^*(\psi_i)}(Y) = \sigma_i \quad \text{and} \quad \mathbf{M}^*(\psi_i)(Y) = y_i \quad (i = 1, 2)$$

with $y_1 \neq y_2$. Then for all $A \in \mathcal{A} \otimes \mathcal{B}$ such that $[A, Y] \leq \delta \|A\|$:

$$\begin{aligned} &|\mathbf{M}^*(|\alpha\psi_1 + \beta\psi_2\rangle\langle\alpha\psi_1 + \beta\psi_2|)(A) - \mathbf{M}^*(|\alpha|^2|\psi_1\rangle\langle\psi_1| + |\beta|^2|\psi_2\rangle\langle\psi_2|)(A)| \\ &\leq \frac{\delta + \sigma_1 + \sigma_2}{|y_1 - y_2|} \|A\|. \end{aligned}$$

Proof:

As the proof of proposition (6), but we now estimate equation (1.14) with lemma (7) instead of lemma (5).

q. e. d.

The ratio $\frac{\sigma_1 + \sigma_2}{|y_1 - y_2|}$ is an indicator of the quality of measurement: suppose you know that, prior to measurement, \mathcal{A} is either in state ψ_1 or ψ_2 . To find out which, you perform measurement. Suppose $y_1 < y_2$, then you conclude that the state was ψ_2 if the pointer Y takes value $\geq \frac{y_1 + y_2}{2}$. The probability of deciding ψ_2 while the state was really ψ_1 is less than $\frac{4\sigma_1^2}{|y_1 - y_2|^2}$ by Chebyshev's inequality: $\mathbb{P}_{\mathbf{M}^*(\psi_1), Y}(|Y - y_1| \geq \frac{|y_1 - y_2|}{2}) \leq \frac{4\sigma_1^2}{|y_1 - y_2|^2}$. Of course the same goes for $1 \leftrightarrow 2$, so that $\frac{\sigma_1}{|y_1 - y_2|} + \frac{\sigma_2}{|y_1 - y_2|}$ indicates the quality of measurement indeed.

Conclusions

For the case of a pointer outside \mathcal{A} , I will summarize some consequences of measurement which will come in handy in the next chapter:

- Collapse takes place on the commutant Y' of the pointer. This includes the original algebra $\mathcal{A} \otimes \mathbb{I}$.
- An imperfect collapse will also occur on observables A commuting well with the pointer Y in the sense that $\|[A, Y]\| \leq \delta \|A\|$ for some small δ .
- If the measurement is imperfect (a $|\psi_i\rangle$ -measurement yields outcome a_i only with high probability), then also an imperfect collapse will occur on the commutant of the pointer.

Chapter 2

Macroscopic Observables

Suppose that an outside observer $\mathcal{C} \subset \mathcal{B}$ performs measurement on a system \mathcal{A} . Then we have seen that a collapse must always take place *on the original system* $\mathcal{A} \otimes \mathbb{I}$. This simple and rigorous law of nature is, I believe, the collapse of the wave function usually alluded to in elementary textbooks on quantum mechanics (e.g. [B&J], [Dir], [Böh], [Neu] and even [Jau, p. 184]).

But on the combined system $\mathcal{A} \otimes \mathcal{B}$ there always remain observables with respect to which no collapse occurs. Indeed, Hepp and later Bell (see [Hep] and [Bel]) have pointed out that all one has to do to track these down is to run time evolution backwards.

On M_2 , the observable σ_x may serve to distinguish ψ_+/ψ_- mixtures from superpositions (p. 6). But if an outside observer $\mathcal{C} \subset \mathcal{B}$ performs a ψ_+/ψ_- measurement $\mathbf{M}^* : \mathcal{S}(M_2) \rightarrow \mathcal{S}(M_2 \otimes B)$ of the form $\mathbf{M}^*(\rho) = \alpha^*(\rho \otimes \tau)$, then

$$\mathbf{M}^*(\rho)(\alpha^{-1}(\sigma_x \otimes \mathbb{I})) = \rho \otimes \tau(\sigma_x \otimes \mathbb{I}) = \rho(\sigma_x)$$

so by performing an $\alpha^{-1}(\sigma_x \otimes \mathbb{I})$ -measurement on $M_2 \otimes \mathcal{B}$, a second observer $\tilde{\mathcal{C}}$, outside M_2 and \mathcal{B} , can indeed ascertain that a full collapse has not taken place¹. In practice however, collapse is observed after measurement, even by the second observer.

In my view (interpretation 3), collapse on closed systems simply does not occur. Ever. Which leaves me to answer:

Question 1 *Why are the remaining coherences so hard to observe in practice?*

Suppose one were to take the point of view that a rigorous collapse on closed systems does occur after measurement. (Interpretation 2.) Then one needs to answer the following question:

Question 2 *Exactly when does collapse replace unitary time evolution on closed systems, and why is it so hard, in practice, to see the difference between collapse at one time rather than another?*

Which point of view to take is merely a matter of taste, not of importance. An answer to question (1) entails an answer to the last part of question (2): If it is hard to see the

¹This in contrast to state *reduction*, which involves only one observer. See also p. 60

difference between unitary time evolution and collapse, it is certainly hard to see when the former goes into the latter.

To question (1), I see three answers. Two of them testify to the weirdness of the observables on which no collapse occurs. Or rather, to the occurrence of collapse on classes of ordinary observables:

2.1 Collapse on the Measurement Apparatus

First of all, we have seen how collapse comes about on $\mathcal{A} \otimes \mathbb{I}$: one simply applies lemma (7) to $\phi_1 = \mathbf{M}^*(\psi_1)$ and $\phi_2 = \mathbf{M}^*(\psi_2)$.

Collapse on the Original System

The measurement is perfect if, starting with ψ_i , ($i = 1, 2$), the pointer position after measurement is always $y_i = \mathbf{M}^*(\psi_i)(Y)$: then the corresponding variances σ_i^2 of the pointer observable $Y = \mathbb{I} \otimes \tilde{Y}$ equal 0. This results in a perfect collapse on $\mathcal{A} \otimes \mathbb{I} \subset (\mathbb{I} \otimes \tilde{Y})'$: proposition (6).

Suppose the measurement is flawed, i.e. the input ψ_i , ($i = 1, 2$) does not absolutely guarantee the pointer output y_i . Then it may still be possible to draw reliable conclusions from the pointer about the examined system, provided that $\sigma_i \ll |y_1 - y_2|$ for $i = 1, 2$, or briefly $\frac{\sigma_1 + \sigma_2}{|y_1 - y_2|} \ll 1$.

We no longer have any reason to expect the ‘clean’ collapse discussed above, but still an imperfect measurement must surely induce some imperfect collapse on $\mathcal{A} \otimes \mathbb{I} \subset Y'$. This is proposition (6).

Collapse on the Measurement Apparatus

But a wider range of collapse can be obtained with the same ease: assume for example that a measurement \mathbf{M}^* distinguishes two eigenstates ψ_{x_1} and ψ_{x_2} of some $X \in \mathcal{A}$ in a *repeatable* fashion. This means that another measurement of $X \otimes \mathbb{I}$ (perhaps by another observer) in state $\mathbf{M}^*(\psi_{x_i})$, ($i = 1, 2$) will once again yield x_i with certainty:

$$\mathbf{M}^*(\psi_{x_i})(X \otimes \mathbb{I}) = x_i \quad \text{and} \quad \mathbf{Var}_{\mathbf{M}^*(\psi_{x_i})}(X \otimes \mathbb{I}) = 0 \quad (i = 1, 2).$$

Then we can apply proposition (6) to the measured observable $X \otimes \mathbb{I}$ instead of the pointer $\mathbb{I} \otimes \tilde{Y}$. Collapse then occurs not only on $(\mathbb{I} \otimes \tilde{Y})'$, but also on $(X \otimes \mathbb{I})'$, which includes the algebra of the measurement apparatus, $\mathbb{I} \otimes \mathcal{B}$.

In exactly the same manner as above, an approximate state collapse on $(X \otimes \mathbb{I})' \supset \mathbb{I} \otimes \mathcal{B}$ follows from proposition (8), provided that

$$\mathbf{Var}_{\mathbf{M}^*(\psi_{x_1})}(X \otimes \mathbb{I}) + \mathbf{Var}_{\mathbf{M}^*(\psi_{x_2})}(X \otimes \mathbb{I}) \ll |x_1 - x_2|^2.$$

So the thoroughness of collapse on $(X \otimes \mathbb{I})'$ is not regulated by the quality of measurement, but by how well $\mathbf{M}^*(\psi_i)$, ($i = 1, 2$) remain eigenstates of X . Already, we have a first answer to question (1):

Answer 1 *In a repeatable measurement, the remaining coherence can neither be detected on the original system, nor on the measurement apparatus alone.*

2.2 Collapse on Local and Global Observables

We have seen that there *are* observables on which no collapse occurs, but if the measurement is repeatable they lie neither in the original system $\mathcal{A} \otimes \mathbb{I}$ nor in the measurement apparatus $\mathbb{I} \otimes \mathcal{B}$. Moreover, the vector space (Not the algebra!) spanned by the commutant of the pointer Y and that of the measured observable X , denoted $X' + Y'$, allows no coherences to be detected. This already testifies to the weirdness of the observables we are looking for: I for one would be very interested to learn about actual measurements performed (with the help of a second measurement apparatus) on observables in $\mathcal{A} \otimes \mathcal{B}$, but outside $X' + Y'$. If they do exist, they are certainly quite exotic.

But there are two more classes of ordinary observables on which collapse takes place: the local ones and the macroscopic ones. This, I believe to be the main point of Klaus Hepp's 1972 article² 'Quantum Theory of Measurement and Macroscopic Observables' [Hep].

2.2.1 K. Hepp: Quasilocal Algebras

Hepp investigates the possibility of modelling time evolution by a weak limit of automorphisms and, as the title suggests, pointers by so called 'macroscopic observables' in a quasilocal algebra.

Let me try to suppress my sense of guilt about not explaining these notions properly by giving an example: imagine a countably infinite chain of quantum spins M_2 indexed by $n \in \mathbb{N}$, their position on the real line. Local observables are supposed to affect only a finite amount of spins. For example, the spin in the z -direction of atom number i , $\sigma_z^i \stackrel{\text{def}}{=} \mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \sigma_z \otimes \mathbb{I} \otimes \mathbb{I} \otimes \dots$, is a local observable. So is $\frac{1}{N} \sum_{i=1}^N \sigma_z^i$, the average z -spin over the first N atoms. Now a quasilocal observable is almost local in the sense that, outside a finite amount of sites, it is arbitrarily close to \mathbb{I} in norm.

Macroscopic observables however are *not* supposed to lie in the quasilocal algebra. We would like them to be something like 'averages': let Y_n be a uniformly bounded sequence of local observables 'converging to infinity' in the sense that Y_n has to do with spins arbitrarily far away from the origin for n sufficiently large. Then it would be pleasant to call $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^N Y_i$ a macroscopic observable. For example, take $Y_n = \sigma_z^n$. Then $S_z = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=0}^N \sigma_z^i$ is the average spin in the z -direction.

Unfortunately, this limit does not exist. At least not in norm. But if we choose one particular state on the algebra, we may form its GNS-representation. (See [K&R, p. 278].) Then we have at our disposal a weak topology, coarser than the norm topology, in which the limit may well exist.

In short, macroscopic observables lie in the weak closure of some represented quasilocal algebra, but not in the algebra itself. The crux of Hepp's article is macroscopic difference:

²The author himself did not, for as far as I can tell, seek to make this particular point. I take the liberty of interpreting his results in a different fashion, utilizing Hepp's considerable mathematical achievements in a context slightly different from the one originally intended. The following digression should not be seen as a summary of [Hep], but as a highly personal interpretation.

Definition 3 (Macroscopic Difference) Let ω_1 and ω_2 be states on a quasilocal algebra \mathcal{D} . Then ω_1 and ω_2 are called macroscopically different if there exists a uniformly bounded sequence of observables Y_n converging to infinity, and real numbers $y_1 \neq y_2$ such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{n=N} \omega_i(Y_n) = y_i \quad (i = 1, 2).$$

and the upshot is formed by the following two lemmas:

Lemma 9 (Lemma 6 of [Hep]) Let ω_1 and ω_2 be macroscopically different states on a quasilocal algebra \mathcal{D} having short range correlations. Then ω_1 and ω_2 are disjoint.

Lemma 10 (Lemma 3 of [Hep]) Consider two disjoint states ω_1 and ω_2 on a quasilocal algebra \mathcal{D} , and two (not necessarily disjoint) sequences $\omega_{1,t}$ and $\omega_{2,t}$ such that $\lim_{t \rightarrow \infty} \omega_{i,t} = \omega_i$ ($i = 1, 2$). Let $(\pi_t, \mathcal{H}_{\pi_t})$ be representations of \mathcal{D} and $\psi_{1,t}, \psi_{2,t} \in \mathcal{H}_{\pi_t}$ such that $\omega_{i,t}(A) = \langle \psi_{i,t} | \pi_t(A) | \psi_{i,t} \rangle$ ($i = 1, 2$) for all $A \in \mathcal{D}$. Then for all quasilocal $D \in \mathcal{D}$:

$$\lim_{t \rightarrow \infty} \langle \psi_{1,t} | \pi_t(D) | \psi_{2,t} \rangle = 0.$$

These lemmas may be used as follows: we are attempting to measure, say, the observable σ_z in the algebra M_2 . We do this by coupling M_2 to a quasilocal algebra \mathcal{B} in a state τ . We now seek automorphisms α_t of $\mathcal{D} = M_2 \otimes \mathcal{B}$ such that $\lim_{t \rightarrow \infty} \alpha_t^*(\psi_i \otimes \tau) = \omega_i$ ($i = 1, 2$), where ω_1 and ω_2 are short-range correlated, macroscopically different states on the quasilocal algebra \mathcal{D} . (Hepp gives several explicit examples of such constructions.)

We may now use lemmas (9) and (10) consecutively to see that for each fixed quasilocal $A \in M_2 \otimes \mathcal{B}$, all the ‘cross-terms’ go to zero:

$$\lim_{t \rightarrow \infty} \langle \psi_{1,t} | \pi_t(D) | \psi_{2,t} \rangle = 0$$

in the sense of lemma (10), with $\omega_{i,t} = \alpha_t^*(\psi_i \otimes \tau)$ for $i = 1, 2$.

The relevance of this all to question (1) is clear:

Pointers used in real life are often macroscopic.

Furthermore, macroscopic information is more easily detected than microscopic information. This is the very reason for using macroscopic pointers. A ray of light shining onto a measurement apparatus is almost certain to record the (macroscopic) position of the pointer, i.e. the average position of some 10^{23} atoms. It may even accidentally record the position of one single atom. But it is very unlikely to record the detailed excitation of *each* of these atoms from their respective equilibrium positions.

In other words: Hepp points out the classes of quasilocal and macroscopic observables as ordinary ones. In his spirit, an answer to question (1) could be:

‘In the course of measurement with a macroscopic pointer, reduction occurs increasingly well on all quasilocal observables’

However, it is not easy to pinpoint the exact physical relevance of Hepp’s weak-operator limit procedure:

- First of all, the notion of a 'Macroscopic' observable is only mathematically defined on a quasilocal algebra. In reality, the algebra describing an actual measurement apparatus is usually extremely large, but not quasilocal. In what way, if at all, are Hepp's result approximately valid?
- Secondly, we only come 'close' to macroscopically disjoint states in the weak topology. At each fixed time t , α_t is still automorphic. So there remain quasilocal observables A for which the cross-terms are large. On the other hand, for each fixed quasilocal observable A , the cross-terms do become small in the course of time. Putting it more precisely and less clearly:

$$\forall \epsilon > 0 \quad \forall A \in \mathcal{A} \otimes \mathcal{B} \quad \exists t \in \mathbb{R} : \quad t' \geq t \Rightarrow |\langle \psi_{+,t} | \pi_t(A) | \psi_{-,t} \rangle| \leq \epsilon ,$$

yet

$$\exists \epsilon > 0 \quad \forall t \in \mathbb{R} \quad \exists A \in \mathcal{A} \otimes \mathcal{B} \quad \exists t' \geq t : \quad |\langle \psi_{+,t} | \pi_t(A) | \psi_{-,t} \rangle| > \epsilon .$$

2.2.2 Local Algebras

We'll give up quasilocal algebras all together, and with it the sharp distinction between local and macroscopic observables. Then we will give estimates on the amount of state collapse on 'local' observables, yet to be defined, based on:

- Exactly how local the observable is.
- Exactly how macroscopic the pointer is.
- Exactly how much macroscopic difference there is.

Instead of utilizing Hepp's machinery, we shall resort to lemma (7). Let us describe our combined system \mathcal{D} by a large but finite number N of possibly different atoms, each described by an algebra \mathcal{D}_i : $\mathcal{D} = \bigotimes_{i=1}^N \mathcal{D}_i$. Such an algebra, plus its (non-unique) subdivision into atoms³ may be called a local algebra.

Let $X^i \in \mathcal{D}_i$. We will denote by X_i the observable $\mathbb{I}_1 \otimes \dots \otimes \mathbb{I}_{i-1} \otimes X^i \otimes \mathbb{I}_{i+1} \otimes \dots \otimes \mathbb{I}_N$ in \mathcal{D} . Although no sharp distinction can be made between macroscopic and microscopic observables, it is intuitively clear that for each set $\{X^i | i \in \{1, \dots, N\}, X^i \in \mathcal{D}_i\}$:

- $\frac{1}{N} \sum_{i=1}^N X_i$ is an average, or very global observable. It might well be observed by accident.
- X_{37} is a very local observable, representing detailed information about one particular atom. (Number 37.) A measurement of X_{37} would probably cost a lot of effort, and is unlikely to be performed by accident.
- $X^1 \otimes X^2 \otimes \dots \otimes X^{N-1} \otimes X^N$ represents an observable giving detailed information about *all* atoms in the measurement apparatus. It is unlikely that such a measurement can ever be performed at all, let alone accidentally.

³By 'atom', I just mean some small part of the algebra. It may represent an electron, atom or molecule, or any other structure small compared to \mathcal{D} . The same algebra $\mathcal{D} = \bigotimes_{i=1}^{6 \times 10^{23}} (\mathcal{D}_H \otimes \mathcal{D}_O \otimes \mathcal{D}_H)$, describing a mole of water, must be considered a different local algebra according to whether one chooses the hydrogen and oxygen atoms \mathcal{D}_H and \mathcal{D}_O as local atoms, or the water molecules $\mathcal{D}_H \otimes \mathcal{D}_O \otimes \mathcal{D}_H$.

So we would like to show that, if the pointer is $Y = \frac{1}{N} \sum_{i=1}^N X_i$, some very global observable, then observables with respect to which no collapse takes place are certainly not very global, nor very local, and typically of the third variety mentioned above. We will start by quantifying these rather vague notions:

- **Definition 4 (n -local)** A Hermitean X is called n -local iff there exist integers $1 \leq i_1 < \dots < i_n \leq N$ such that $X \in \mathcal{D}_{i_1} \otimes \dots \otimes \mathcal{D}_{i_n} \subset \mathcal{D}$.

An observable $X \in \mathcal{D}$ is ‘ n -local’ if it only affects n atoms. X_{37} is 1-local. Of course, each n -local observable is also \tilde{n} -local if $\tilde{n} \geq n$.

- **Definition 5 (κ -global)** A Hermitean $Y \in \mathcal{D}$ is called κ -global iff there exist $M \in \mathbb{N}$, $1 \leq i_1 < \dots < i_M \leq N$ and Hermitean $Y^{i_k} \in \mathcal{D}_{i_k}$ such that:

- $Y = \frac{1}{M} \sum_{k=1}^M Y^{i_k}$.
- $\kappa \geq \frac{\|Y^{i_k}\|}{M}$ for all k .

For example, $\frac{1}{N} \sum_{i=1}^N \sigma_z^i$ as on page 27 is N^{-1} -global. X_{37} is $\|X_{37}\|$ -global, for example because it equals $X_{37} = \frac{1}{1}(X_{37})$, or, if you happen to be in a troublesome mood, because $X_{37} = \frac{1}{N}(0 + \dots + 0 + NX_{37} + 0 \dots + 0)$. An observable will be called global if it is κ -global for some $\kappa \in \mathbb{R}$. Not all observables are global.

- The amount of difference between two states ϕ_1 and ϕ_2 on some global Hermitean Y can easily be quantified by the ratio $\frac{\sigma_1 + \sigma_2}{|y_1 - y_2|}$, where $y_i = \phi_i(Y)$ and $\sigma_i^2 = \phi_i(Y^2) - \phi_i(Y)^2$, ($i = 1, 2$).

These definitions allow us to apply lemma (7) in the situation of a measurement using a global pointer to distinguish ψ_i from ψ_j : $\mathbf{M}^*(\psi_i)$ and $\mathbf{M}^*(\psi_j)$ are globally different.

Corollary 7.1⁴ Let $\phi_1, \phi_2 \in \mathcal{S}(\mathcal{D})$ be κ -globally different vector states, i.e. there is a κ -global Hermitean $Y \in \mathcal{D}$ such that $\phi_1(Y) \neq \phi_2(Y)$. Let $y_{1,2} = \phi_{1,2}(Y)$ be the expectation of Y in $\phi_{1,2}$, and $\sigma_{1,2}^2 = \mathbf{Var}_{\phi_{1,2}}(Y)$ the variance. Let $\alpha, \beta \in \mathbb{C}$ be such that $|\alpha|^2 + |\beta|^2 = 1$. Then, for every n -local A :

$$|\langle \alpha\phi_1 + \beta\phi_2 | A | \alpha\phi_1 + \beta\phi_2 \rangle - (|\alpha|^2 \langle \phi_1 | A | \phi_1 \rangle + |\beta|^2 \langle \phi_2 | A | \phi_2 \rangle)| \leq \frac{2n\kappa + \sigma_1 + \sigma_2}{|y_1 - y_2|} \|A\|.$$

And for every global observable $Y' = \frac{1}{M'} \sum_{i=1}^{M'} Y_{j_i}$ with $\|Y_{j_i}\| \leq y'$:

$$|\langle \alpha\phi_1 + \beta\phi_2 | Y' | \alpha\phi_1 + \beta\phi_2 \rangle - (|\alpha|^2 \langle \phi_1 | Y' | \phi_1 \rangle + |\beta|^2 \langle \phi_2 | Y' | \phi_2 \rangle)| \leq \frac{2\kappa + \sigma_1 + \sigma_2}{|y_1 - y_2|} y'.$$

⁴Consider the example of a cloud of N particles, with positions x^i and momenta p^i . These observables are not bounded, but since $[x^i, p^j]$ is, this is merely a technical problem which may be averted by, for example, a cut-off. The position of the cloud, $X = \frac{1}{N} \sum_{i=1}^N x_i$, is increasingly macroscopic for increasing N . However, the *total* momentum of the cloud, $P = \sum_{i=1}^N p_i$, is $\|p^i\|$ -global, irrespective of N . So since $[X, P] = i\hbar$, it would seem that reduction on X , using P as pointer, is good because \hbar is small, and not because N is large. This is not the case however: of importance is the ratio $\hbar/|p_1 - p_2|$, and typical momentum differences *do* grow as N increases.

Proof:

Since $2|\alpha||\beta| \leq 1$, all we have to do is apply lemma (7) to $\langle \phi_1 | A | \phi_2 \rangle$. For the first inequality, we write $Y = \frac{1}{M} \sum_{k=1}^M Y_{i_k}$. Since at most n of the Y_{i_k} do not commute with A , we have $\|[Y, A]\| = \|\frac{1}{M} \sum_{k=1}^M [Y_{i_k}, A]\| \leq 2n\|A\| \frac{\max_k \|Y_{i_k}\|}{M} \leq 2n\kappa\|A\|$. For the second inequality, we use that $\|[Y_{i_k}, Y_{j_l}]\| \leq 2\kappa M y'$, and that it equals zero if $i_k \neq j_l$. We obtain $\|[Y, Y']\| = \|\frac{1}{M'M} \sum_{k=1, l=1}^{k=M, l=M'} [Y_{i_k}, Y_{j_l}]\| \leq \frac{1}{M'M} \sum_{k=1, l=1}^{k=M, l=M'} \delta(i_k, j_l) 2\kappa M y' \leq 2\kappa y'$. Of course $\|Y'\| \leq y'$.

q.e.d.

Bear in mind that κ , for a typical pointer, will have values in the order of 10^{-23} . For a perfect measurement ($\sigma_{1,2} = 0$) this means one may probe detailed information about billions and billions of atoms simultaneously without running the slightest risk of encountering any lack of collapse.

Furthermore, in the case of non-perfect measurement using a κ -global pointer, as long as $n\kappa \ll \sigma_1 + \sigma_2$, all experimentally observed coherence may be attributed to the poor quality of measurement. All of this constitutes a second answer to question (1):

Answer 2 *When using a very global pointer, the coherence remaining after measurement can neither be detected on very global observables, nor on very local ones.*

2.3 Global Information Leakage

So suppose we are in the circumstance of a repeatable measurement with the help of a very global ‘pointer’, say an actual pointer. We have seen that observables on which no collapse occurs are present on the combined system, but they can neither be entirely inside the measurement-apparatus, nor entirely in the atom. They can also not be very global, nor very local. All in all, they are pretty weird indeed. Yet in principle, they *do* exist and they *can* be measured by a second observer.

2.3.1 Information Leakage After Measurement

One should be extremely careful with this kind of reasoning, however. Imagine, for example, that M_2 represents a two-level atom, and \mathcal{B} describes some large measuring apparatus, measuring eigenstates of $\sigma_z \otimes \mathbb{I}$ with pointer $\mathbb{I} \otimes Y$ which represents, literally, the position of a pointer. (The average position of all atoms in the pointer is of course rather global.) Then as soon as $\mathbb{I} \otimes Y$ is measured (with the help of an ancillary system, e.g. light reflecting on the pointer and reaching our eyes), collapse on the combined atom-apparatus system takes place. It is of course immaterial whether or not someone is actually *looking* at the photons. If even the smallest speck of light were to fall on the pointer, the information about the pointer position would already be encoded in the light, causing full collapse on the atom-apparatus system.

So as soon as the information about the pointer position has reached our eyes, we can be mathematically certain that collapse on the atom-apparatus-system has taken place, even on those extraordinary observables that commute poorly with the pointer of the measurement apparatus. However, if no such information has reached our eyes, we may still be practically sure that all measurement on the combined atom-apparatus will reveal that collapse has taken place, unless extreme measures (e.g. shielding, extreme cooling) have been taken to prevent pointer-information from leaking out of the system. A similar line of reasoning may provide a third answer to question (1), again due to lemma (7):

Answer 3 *If information leaks from the pointer into the outside world, collapse unavoidably takes place on the combination of system and measurement apparatus. In practice, global pointers constantly leak information.*

2.3.2 Information Leakage in General

A Macroscopic system may be modelled by some local algebra $\mathcal{D} = \bigotimes_{i=1}^N \mathcal{D}_i$. If this system is interacting normally with the outside world, (the occasional photon happens to scatter on it, for instance) then one may imagine a number of very global observables being measured continually, with a certain measurement uncertainty⁵ σ . For a decent definition of this ‘measurement uncertainty’ or ‘quality’ σ , I will have to refer to chapter 3, definitions (11) and (12). But it entails the σ_1 and σ_2 of lemma (7) being $\leq \sigma$ for eigenstates of X .

⁵According to proposition (15), simultaneous perfect measurement is not possible, but global observables normally commute well enough for the limits of accuracy imposed by proposition (16) to remain well below the macroscopic scale.

This enables us to apply lemma (7). It tells us that all coherences between eigenstates ψ_{x_1} and ψ_{x_2} of global observables X are continually vanishing on all of \mathcal{D} , (the pointer, e.g. a beam of light, is outside the system), provided their eigenvalues x_1 and x_2 satisfy $|x_1 - x_2| \gg 2\sigma$.

Take for example a collection of N spins, $\mathcal{D} = \bigotimes_{i=1}^N M_2$. Suppose that for $\alpha = x, y, z$, the observables $S_\alpha = \frac{1}{N} \sum_{i=1}^N \sigma_\alpha^i$ are continually being measured with an accuracy $N^{-1} \ll \sigma \ll 1$. For N in the order of Avogadro's number, $N \sim 6 \times 10^{23}$, this allows for extremely accurate measurement. Then between globally different eigenstates of S_α , i.e. states for which the eigenvalues $|s_\alpha - \tilde{s}_\alpha| \gg \sigma$, coherences are constantly disappearing. However, the measurement need not have any effect⁶ on states which only differ on a small scale. Take for instance $\rho \otimes |+\rangle$ and $\rho \otimes |-\rangle$, with ρ some state on $N - 1$ spins. Indeed, $|s_\alpha - \tilde{s}_\alpha| \leq 2/N \ll \sigma$, so lemma (7) is vacuous in this case.

We see how σ produces a smooth boundary between the macroscopic and the microscopic world: global processes (involving S_α -differences $\gg \sigma$) continually suffer from loss of coherence, while local processes (involving S_α -differences $\ll \sigma$) are unaffected.

⁶Of course it is *possible* for a measurement to destroy coherence between $\rho \otimes |+\rangle$ and $\rho \otimes |-\rangle$. Just take any old measurement, and add a 'decoherence operation' by hand. The net result is a measurement destroying coherence.

2.4 Conclusion

In order for a second observer to notice the lack of collapse after a first observer has measured $X \in \mathcal{A}$ with a global pointer, he or she must do the following:

- Keep the original system \mathcal{A} and the system \mathcal{B} , containing the first observer, from interacting with the outside world, for instance by shielding or extreme cooling.
- Then perform a measurement on an observable $\tilde{X} \in \mathcal{A} \otimes \mathcal{B}$ which does not commute with the global pointer: it cannot be in the original system \mathcal{A} , it cannot be very global and it cannot be very local. If the first measurement is repeatable, it cannot lie entirely in \mathcal{B} either.
- The outcome of the first measurement must remain unknown to the second observer. Suppose that an observable in the second observer serves as a pointer to the first measurement. The net situation would then be a measurement of $X \in \mathcal{A}$ by an observer outside of $\mathcal{A} \otimes \mathcal{B}$. A second measurement will then always show collapse on $\mathcal{A} \otimes \mathcal{B}$.

Under these circumstances, it cannot be excluded that coherences are experimentally detected by the second observer.

Paradox

All of this leaves us with one glaring paradox: surely the first observer, remembering the outcome of X -measurement, can perform \tilde{X} -measurement and observe that there is no collapse, let alone reduction, contradicting proposition (3)?

The answer is no. The first observer cannot simultaneously measure \tilde{X} *and* remember a value of X . We will elaborate this in proposition (26), where we will have some more tools at our disposal.

Chapter 3

Measurement Inequalities

There are many different concepts of measurement. Up until now, we have investigated mappings $\mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ of the form $\mathbf{M}^*(\rho) = \alpha^*(\rho \otimes \tau)$ for some automorphism α of $\mathcal{A} \otimes \mathcal{B}$ and $\tau \in \mathcal{S}(\mathcal{B})$. But other points of view are possible. For instance, von Neumann (see [Neu]) and Holevo (see [Hol]) define measurement as an affine mapping from $\mathcal{S}(\mathcal{A})$ to $\mathcal{S}(\mathcal{C}(\Omega))$, the space of probability measures on the Borel σ -algebra in $\Omega \subset \mathbb{R}$.

In order to cover all concepts of measurement at the same time, we will investigate completely positive operations. With their help, we will define perfect measurement and we will define unbiased measurement. We will then rigorously define the quality of unbiased measurement¹.

After this, the way will be cleared for quite general statements on the trade-off between measurement quality and the amount of disturbance².

3.1 Completely Positive Operations

What do we expect from *any* physical operation $\mathbf{T}^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{B})$ from one (quantum) probability space into another? We formulate three natural requirements (copied from [Maa]):

- The *stochastic equivalence principle* is common to all interpretations of postulates (1) and (2). It states that a system that is in state ρ_1 with probability λ_1 and in state ρ_2 with probability λ_2 cannot be distinguished from a system in state $\lambda_1\rho_1 + \lambda_2\rho_2$. Therefore, \mathbf{T}^* must be *affine*: for all $0 \leq \lambda_1, \lambda_2 \leq 1$ such that $\lambda_1 + \lambda_2 = 1$,

$$\lambda_1 \mathbf{T}^*(\rho_1) + \lambda_2 \mathbf{T}^*(\rho_2) = \mathbf{T}^*(\lambda_1\rho_1 + \lambda_2\rho_2)$$

¹The attentive reader may have noticed the grotesque ugliness of proposition (4). This results from the ad hoc use of $|\mathbf{M}^*(\rho)(\mathbf{Q}) - \rho(\mathbf{P})|$ as a measure of quality.

²Although propositions and proofs will be different from the ones encountered before, their interpretation will be similar if not the same. In order not to disturb the flow of reasoning, I've chosen to once again go over details exhaustively mentioned before. My apologies to the reader.

\mathbf{T}^* can thus be extended to a linear mapping between the full duals of \mathcal{A} and \mathcal{B} , so that \mathbf{T}^* is the dual of a linear map $\mathbf{T} : \mathcal{B} \rightarrow \mathcal{A}$. This justifies our notation: $\mathbf{T}^*(\rho) = \rho \circ \mathbf{T}$.

- In order for \mathbf{T}^* to map states to states, it must respect normalization and positivity: $\mathbf{T}^*(\rho)(\mathbb{I}) = 1$ and $\mathbf{T}^*(\rho)(B^\dagger B) \geq 0 \quad \forall \quad B \in \mathcal{B}, \quad \rho \in \mathcal{S}(\mathcal{A})$. Equivalently, $\mathbf{T}(\mathbb{I}) = \mathbb{I}$ and $B \geq 0 \Rightarrow \mathbf{T}(B) \geq 0$.
- So \mathbf{T}^* is linear, normalized and positive. But it was realized by K. Krauss in the 1970's that it must be possible to couple \mathcal{A} and \mathcal{B} to another system \mathcal{C} and perform the operation \mathbf{T}^* on \mathcal{A} , leaving \mathcal{C} untouched. This leads us to the last requirement (see [Kra]).

An operation \mathbf{T}^* is called *n-positive* if the map $\mathbf{id}_n^* \otimes \mathbf{T}^* : \mathcal{S}(M_n \otimes \mathcal{B}) \rightarrow \mathcal{S}(M_n \otimes \mathcal{A})$ defined by $\tau \otimes \rho \mapsto \tau \otimes (\mathbf{T}^*(\rho))$ is linear, normalized and positive. An operation is called *completely positive* if it is *n-positive* for all $n \in \mathbb{N}$.

There exist positive operations which are not completely positive. Formulating the above in the Heisenberg picture, we define a linear, continuous map $\mathbf{id}_n \otimes \mathbf{T} : M_n \otimes \mathcal{B} \rightarrow M_n \otimes \mathcal{A}$ by $\mathbf{id}_n \otimes \mathbf{T}(A \otimes B) = A \otimes \mathbf{T}(B)$ for all $A \in M_n$ and $B \in \mathcal{B}$.

Definition 6 (Complete Positivity) *Let \mathcal{A} and \mathcal{B} be unital C^* -algebras. A linear map $\mathcal{B} \rightarrow \mathcal{A}$ is called completely positive if³ $\mathbf{T}(\mathbb{I}) = \mathbb{I}$ and if for all $D \in M_n \otimes \mathcal{B}$, $n \in \mathbb{N}$:*

$$D \geq 0 \quad \implies \quad \mathbf{id}_n \otimes \mathbf{T}(D) \geq 0.$$

The class of completely positive operations was invented to encompass every physical operation you could ever want. For example, it contains all automorphisms, *-homomorphisms and states, as well as dilations to automorphisms. A positive operation from or to an abelian algebra is automatically completely positive, and it hardly needs mentioning that both concepts of measurement mentioned above are completely positive too.

We will proceed to investigate completely positive operations. It is surprising how much can be said about so general an object.

³We assume all completely positive operations to be automatically unital: $\mathbf{T}(\mathbb{I}) = \mathbb{I}$. This is not always so in the literature.

3.2 A Cauchy-Schwarz Inequality

If \mathcal{A} , \mathcal{B} are C*-algebras, one can define a sesquilinear map $\mathbf{F}_{\mathbf{T}} : \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{A}$.

Definition 7 Let $\mathbf{T} : \mathcal{B} \rightarrow \mathcal{A}$ be a 4-positive unital operation. Let $A, B \in \mathcal{B}$. Then

$$\mathbf{F}_{\mathbf{T}}(A, B) \stackrel{\text{def}}{=} \mathbf{T}(A^\dagger B) - \mathbf{T}(A^\dagger)\mathbf{T}(B).$$

If no confusion is possible, we will often omit the subscript. \mathbf{F} is a sesquilinear positive semidefinite \mathcal{A} -valued form on \mathcal{B} , i.e.

- \mathbf{F} is linear in the second argument, anti-linear in the first.
- $\mathbf{F}(A, B)^\dagger = \mathbf{F}(B, A)$ for all $A, B \in \mathcal{B}$.
- $\mathbf{F}(B, B) \geq 0$ as an operator inequality for all $B \in \mathcal{B}$.

The first and second point follow immediately from $\mathbf{T}(A)^\dagger = \mathbf{T}(A^\dagger)$. We will derive this and the third point shortly, along with an \mathcal{A} -valued Cauchy-Schwarz-inequality. There is also a fourth point of interest, clear from the definition:

- $\mathbf{F}(\mathbb{I}, B) = \mathbf{F}(B, \mathbb{I}) = 0 \quad \forall B \in \mathcal{B}$

The likeness of \mathbf{F} to an inner product incites us to introduce a semi-norm on \mathcal{B} :

Definition 8 (T-norm) Let $\mathbf{T} : \mathcal{B} \rightarrow \mathcal{A}$ be a 4-positive unital operation. Let $B \in \mathcal{B}$. Then

$$\|B\|_{\mathbf{T}} \stackrel{\text{def}}{=} \|\sqrt{\mathbf{F}_{\mathbf{T}}(B, B)}\|.$$

$\|B\|_{\mathbf{T}}$ is called the \mathbf{T} -norm of B . Since \mathbf{T} is always a contraction, we have $\|B\|_{\mathbf{T}} \leq \|B\|$. Of course it is possible that $\|B\|_{\mathbf{T}} = 0$ for $B \neq 0$. If, for example, \mathbf{T} happens to be a C*-homomorphism, then $\mathbf{F}_{\mathbf{T}}$ is identically zero and $\|B\|_{\mathbf{T}} = 0$ for all $B \in \mathcal{B}$. $\mathbf{F}_{\mathbf{T}}$ is in many ways a measure of how well \mathbf{T} respects multiplication.

Real and Imaginary Part

Like any element of \mathcal{A} we can split $\mathbf{F}_{\mathbf{T}}(A, B)$ into a Hermitean and an anti-Hermitean part:

$$\mathbf{F}_{\mathbf{T}}(A, B) = \Re \mathbf{F}_{\mathbf{T}}(A, B) + i \Im \mathbf{F}_{\mathbf{T}}(A, B)$$

with

$$\begin{aligned} 2\Re \mathbf{F}_{\mathbf{T}}(A, B) &= \mathbf{T}(A^\dagger B + B^\dagger A) - (\mathbf{T}(A)^\dagger \mathbf{T}(B) + \mathbf{T}(B)^\dagger \mathbf{T}(A)) \\ 2i\Im \mathbf{F}_{\mathbf{T}}(A, B) &= \mathbf{T}(A^\dagger B - B^\dagger A) - (\mathbf{T}(A)^\dagger \mathbf{T}(B) - \mathbf{T}(B)^\dagger \mathbf{T}(A)). \end{aligned}$$

The commutator of A and B is defined by $[A, B] \stackrel{\text{def}}{=} AB - BA$. The anti-commutator by $\{A, B\}_+ \stackrel{\text{def}}{=} AB + BA$. In case of Hermitean A and B the above boils down to:

$$\begin{aligned} 2\Re \mathbf{F}_{\mathbf{T}}(A, B) &= \mathbf{T}(\{A, B\}_+) - \{\mathbf{T}(A), \mathbf{T}(B)\}_+ \\ 2i\Im \mathbf{F}_{\mathbf{T}}(A, B) &= \mathbf{T}([A, B]) - [\mathbf{T}(A), \mathbf{T}(B)]. \end{aligned}$$

On Hermitean A and B , $\Re \mathbf{F}_{\mathbf{T}}(A, B)$ indicates how well \mathbf{T} respects the anti-commutator, while $\Im \mathbf{F}_{\mathbf{T}}(A, B)$ indicates how well it respects the commutator. ($\Im \mathbf{F}_{\mathbf{T}}$ is identically zero on the Hermiteans if and only if \mathbf{T} is a Lie-algebra homomorphism.)

A Cauchy-Schwarz Inequality

This sesquilinear, positive semidefinite $\mathbf{F}_{\mathbf{T}}$ allows for an \mathcal{A} -valued Cauchy-Schwarz⁴ inequality:

Lemma 11 (C*-Cauchy-Schwarz inequality) *Let \mathbf{T} be a four-positive unital operation $\mathcal{B} \rightarrow \mathcal{A}$. Then for all $A, B \in \mathcal{B}$, we have in operator ordering:*

$$\mathbf{F}_{\mathbf{T}}(A, B)\mathbf{F}_{\mathbf{T}}(B, A) \leq \|\mathbf{F}_{\mathbf{T}}(B, B)\|\mathbf{F}_{\mathbf{T}}(A, A).$$

Proof:

First we prove that $\mathbf{T}(A)^\dagger = \mathbf{T}(A^\dagger)$ and that $\mathbf{F}(A, A) \geq 0 \quad \forall A \in \mathcal{B}$. Since \mathbf{T} is two-positive on \mathcal{B} , $\mathbf{id}_2 \otimes \mathbf{T}$ is positive on $M_2 \otimes \mathcal{B}$, and we see that

$$\mathbf{id}_2 \otimes \mathbf{T} \left(\begin{pmatrix} A^\dagger & 0 \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} A & \mathbb{I} \\ 0 & 0 \end{pmatrix} \right) \geq 0$$

or

$$\begin{pmatrix} \mathbf{T}(A^\dagger A) & \mathbf{T}(A^\dagger) \\ \mathbf{T}(A) & \mathbb{I} \end{pmatrix} \geq 0$$

in the operator ordering. In particular it must be Hermitean so that $\mathbf{T}(A)^\dagger = \mathbf{T}(A^\dagger)$. For each $X \geq 0$, also $Y^\dagger X Y \geq 0$ for any Y . Making a convenient choice for Y :

$$Y = \begin{pmatrix} \mathbb{I} & 0 \\ -\mathbf{T}(A) & 0 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} \mathbf{T}(A^\dagger A) - \mathbf{T}(A)^\dagger \mathbf{T}(A) & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

so that $\mathbf{F}(A, A) \geq 0$ for all two-positive \mathbf{T} . Now since \mathbf{T} is four-positive on \mathcal{B} , $\mathbf{id}_2 \otimes \mathbf{T}$ is again two-positive on $M_2 \otimes \mathcal{B}$. Consequently, making a convenient choice of ‘ A ’ in $M_2 \otimes \mathcal{B}$:

$$\mathbf{F}_{\mathbf{id}_2 \otimes \mathbf{T}} \left(\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \right) \geq 0 \quad \forall \quad A, B \in \mathcal{B}.$$

Working out this expression explicitly:

$$\begin{aligned} \mathbf{F}_{\mathbf{id}_2 \otimes \mathbf{T}} \left(\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \right) &= \\ &= \mathbf{id}_2 \otimes \mathbf{T} \left(\begin{pmatrix} A^\dagger & 0 \\ B^\dagger & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \right) - \\ &\quad \mathbf{id}_2 \otimes \mathbf{T} \left(\begin{pmatrix} A^\dagger & 0 \\ B^\dagger & 0 \end{pmatrix} \right) \mathbf{id}_2 \otimes \mathbf{T} \left(\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \right) \end{aligned}$$

⁴Reminiscent of the Cauchy-Schwarz inequality for Hilbert C*-modules (see [Lan]), but not quite the same: the result is identical, but the conditions differ.

$$\begin{aligned}
&= \begin{pmatrix} \mathbf{T}(A^\dagger A) & \mathbf{T}(A^\dagger B) \\ \mathbf{T}(B^\dagger A) & \mathbf{T}(B^\dagger B) \end{pmatrix} - \begin{pmatrix} \mathbf{T}(A^\dagger)\mathbf{T}(A) & \mathbf{T}(A^\dagger)\mathbf{T}(B) \\ \mathbf{T}(B^\dagger)\mathbf{T}(A) & \mathbf{T}(B^\dagger)\mathbf{T}(B) \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{F}(A, A) & \mathbf{F}(A, B) \\ \mathbf{F}(B, A) & \mathbf{F}(B, B) \end{pmatrix} \\
&\geq 0.
\end{aligned}$$

Once again using $X \geq 0 \Rightarrow Y^\dagger XY \geq 0$, this time with

$$Y = \begin{pmatrix} \mathbb{I} & 0 \\ -\mathbf{F}(B, A) & 0 \end{pmatrix}$$

we obtain, proceeding as above:

$$\mathbf{F}(A, A) - 2\mathbf{F}(A, B)\mathbf{F}(B, A) + \mathbf{F}(A, B)\mathbf{F}(B, B)\mathbf{F}(B, A) \geq 0. \quad (3.1)$$

In the case that $\mathbf{F}(B, B) = 0$, we need to prove that $\mathbf{F}(A, B)\mathbf{F}(B, A) = 0$. Now $\mathbf{F}(NB, NB) = 0$ for $N \in \mathbb{N}$. Applying inequality 3.1 to A and NB , we see that

$$N^2\mathbf{F}(A, B)\mathbf{F}(B, A) \leq \mathbf{F}(A, A)$$

for all N , so that $\mathbf{F}(A, B)\mathbf{F}(B, A) = 0$. In the case that $\mathbf{F}(B, B) \neq 0$, we rephrase inequality 3.1 as

$$\mathbf{F}(A, B)\mathbf{F}(B, A) \leq \mathbf{F}(A, A) + \mathbf{F}(A, B)(\mathbf{F}(B, B) - \mathbb{I})\mathbf{F}(B, A).$$

So, putting $B' = B/\|\sqrt{\mathbf{F}(B, B)}\|$ and noting $\mathbf{F}(B', B') - \mathbb{I} \leq 0$:

$$\frac{\mathbf{F}(A, B)\mathbf{F}(B, A)}{\|\mathbf{F}(B, B)\|} \leq \mathbf{F}(A, A) + \mathbf{F}(A, B')(\mathbf{F}(B', B') - \mathbb{I})\mathbf{F}(B', A) \leq \mathbf{F}(A, A)$$

yielding the required expression.

q.e.d.

Covariance and Uncertainty

We shall give a few easy corollaries to clarify the nature of \mathbf{F} . First of all, it resembles the covariance of a state. Classically, a probability distribution \mathbb{P} on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ induces a covariance on pairs of random variables \mathbf{a}, \mathbf{b} :

$$\mathbf{Cov}_{\mathbb{P}}(\mathbf{a}, \mathbf{b}) = \mathbb{E}_{\mathbb{P}}(\mathbf{a}\mathbf{b}) - \mathbb{E}_{\mathbb{P}}(\mathbf{a})\mathbb{E}_{\mathbb{P}}(\mathbf{b}). \quad (3.2)$$

Where $\mathbb{E}_{\mathbb{P}}$ is the expectation with respect to \mathbb{P} . In a quantum probability space, observables are not represented by random variables, but by Hermitean elements of a C^* -algebra. If one chooses to represent the product observable of A and B by $(AB + BA)/2$, one can generalize (3.2) to arbitrary C^* -algebras.

Definition 9 (Covariance) *Let $\rho \in \mathcal{S}(\mathcal{B})$. Let $A, B \in \mathcal{B}$ Hermitean. Then the covariance of A and B in ρ is defined by:*

$$\mathbf{cov}_{\rho}(A, B) = \rho\left(\frac{1}{2}(AB + BA)\right) - \rho(A)\rho(B).$$

Of course there is no conflict with proposition (2) for commuting A and B . From the C*-Cauchy-Schwarz inequality, we now have two easy corollaries. The first is a standard result, known as the ‘covariance inequality’:

Corollary 11.1 (Covariance Inequality) *Let $\rho \in \mathcal{S}(\mathcal{B})$. Then for all Hermitean $A, B \in \mathcal{B}$:*

$$|\mathbf{cov}_\rho(A, B)|^2 \leq \mathbf{var}_\rho(A) \mathbf{var}_\rho(B).$$

The second standard result is known as the ‘Heisenberg uncertainty relation’:

Corollary 11.2 (Heisenberg Inequality) *Let $\rho \in \mathcal{S}(\mathcal{B})$. Then for all Hermitean $A, B \in \mathcal{B}$:*

$$\left| \rho \left(\frac{[A, B]}{2i} \right) \right|^2 \leq \mathbf{var}_\rho(A) \mathbf{var}_\rho(B).$$

In particular, if $A = i\hbar\partial_x$ and $B = x$, It follows⁵ that $\sigma_A\sigma_B \geq \hbar/2$. We prove both corollaries at the same time:

Proof:

A state ρ on \mathcal{B} is just a completely positive map $\mathcal{B} \rightarrow \mathbb{C}$. So we can form \mathbf{F}_ρ and note that

$$\Re \mathbf{F}_\rho(A, B) = \frac{1}{2}(\mathbf{F}_\rho(A, B) + \mathbf{F}_\rho(B, A)) = \mathbf{cov}_\rho(A, B),$$

$$\Im \mathbf{F}_\rho(A, B) = \frac{1}{2i}(\mathbf{F}_\rho(A, B) - \mathbf{F}_\rho(B, A)) = \rho \left(\frac{[A, B]}{2i} \right).$$

Therefore, both $|\mathbf{cov}_\rho(A, B)|^2 \leq |\mathbf{F}_\rho(A, B)|^2$ and $|\rho(\frac{[A, B]}{2i})|^2 \leq |\mathbf{F}_\rho(A, B)|^2$.

The two corollaries above now follow from the C*-Cauchy-Schwarz inequality:

$$|\mathbf{F}_\rho(A, B)|^2 \leq |\mathbf{F}_\rho(A, A)| |\mathbf{F}_\rho(B, B)| = \mathbf{var}_\rho(A) \mathbf{var}_\rho(B).$$

q.e.d.

In words, corollary (11.1) is the real part of the C*-Cauchy-Schwarz inequality, corollary (11.2) its imaginary part.

Multiplication Theorems

Up to this point, we’ve used a state ρ to construct \mathbf{F}_ρ , but we will encounter \mathbf{F} descendant from more general positive operations $\mathcal{B} \rightarrow \mathcal{A}$ later on. The C*-Cauchy-Schwarz inequality was inspired by a ‘multiplication theorem’ due to R. Werner (see [Wer]):

Corollary 11.3 (Multiplication Theorem) *Let \mathbf{T} be a four-positive unital operation $\mathcal{B} \rightarrow \mathcal{A}$. Let $B \in \mathcal{B}$ such that $\|B\|_{\mathbf{T}} = 0$. Then for all $A \in \mathcal{B}$:*

$$\mathbf{F}_{\mathbf{T}}(A, B) = \mathbf{F}_{\mathbf{T}}(B, A) = 0$$

$$\text{i.e. } \mathbf{T}(A^\dagger B) = \mathbf{T}(A)^\dagger \mathbf{T}(B) \quad \text{and} \quad \mathbf{T}(B^\dagger A) = \mathbf{T}(B)^\dagger \mathbf{T}(A).$$

⁵Since ∂_x and x are not bounded, we are not allowed to apply the C*-Cauchy-Schwarz inequality directly. The statement is true nonetheless.

The proof is immediate from the following generalization, the ‘almost multiplication theorem’.

Corollary 11.4 *Let \mathbf{T} be a four-positive unital operation $\mathcal{B} \rightarrow \mathcal{A}$. Let $B \in \mathcal{B}$. Then for all $A \in \mathcal{B}$:*

$$\|\mathbf{F}_{\mathbf{T}}(A, B)\| \leq \|A\| \|B\|_{\mathbf{T}}.$$

Proof:

For any $A, B \in \mathcal{B}$ we have by the C*-Cauchy-Schwarz inequality $\mathbf{F}_{\mathbf{T}}(A, B)\mathbf{F}_{\mathbf{T}}(B, A) \leq \mathbf{F}_{\mathbf{T}}(A, A)\|B\|_{\mathbf{T}}^2$, so certainly $\|\mathbf{F}_{\mathbf{T}}(A, B)\| \leq \|A\|_{\mathbf{T}}\|B\|_{\mathbf{T}}$. But since $0 \leq \mathbf{F}(A, A) \leq \mathbf{T}(A^\dagger A) \leq \|A\|^2 \mathbb{I}$, we also have $\|A\|_{\mathbf{T}} \leq \|A\|$. The corollary follows.

q.e.d.

This is the form of the C*-Cauchy-Schwarz inequality we will utilize most often.

3.3 Quantum Measurement

With the help of the C*-Cauchy-Schwarz inequality we will investigate quantum measurement. But we will first define it.

3.3.1 Introduction

We will distinguish perfect and unbiased measurement. The former is a special case of the latter.

Perfect Measurement

In order for an operation to be a measurement, it must transport information from X , the observable to be measured, to Y , the pointer-observable. Observation of Y in state $\mathbf{M}^*(\rho)$ must be equivalent to observation of X in state ρ :

Definition 10 (Perfect Measurement) *Let $X \in \mathcal{A}$, $Y \in \mathcal{B}$ be Hermitean. A perfect measurement of X with pointer Y is by definition a completely positive map $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ such that*

$$\mathbb{P}_{\mathbf{M}^*(\rho), Y} = \mathbb{P}_{\rho, X} \quad \forall \rho \in \mathcal{S}(\mathcal{A}).$$

In the Heisenberg picture, this makes $\mathbf{M}_{\mathcal{C}(Y)}$ an injective *-homomorphism:

Proposition 12 *Let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be completely positive. Let $X \in \mathcal{A}$, $Y \in \mathcal{B}$ be Hermitean. Then \mathbf{M} is a perfect measurement of X with pointer Y if and only if*

$$\mathbf{Spec}(X) = \mathbf{Spec}(Y) \quad \text{and} \quad \mathbf{M}(f(Y)) = f(X) \quad \forall f \in \mathcal{C}(\mathbf{Spec}(Y)).$$

Proof:

If $\mathbb{P}_{\rho, X} = \mathbb{P}_{\mathbf{M}^*(\rho), Y}$, then they certainly live on the same measure-space: $\mathbf{Spec}(X) = \mathbf{Spec}(Y)$. By the proof of proposition (1), $\mathbb{P}_{\rho, X} = \mathbb{P}_{\mathbf{M}^*(\rho), Y}$ iff their expectation values on $f \in \mathcal{C}(\mathbf{Spec}(Y))$ are the same. This is so for all $\rho \in \mathcal{S}(\mathcal{A})$ iff $\rho(\mathbf{M}(f(Y))) = \rho(f(X)) \quad \forall \rho \in \mathcal{S}(\mathcal{A})$, or equivalently, iff $\mathbf{M}(f(Y)) = f(X)$.

q.e.d.

In particular, a pointer Y measures only one $X = \mathbf{M}(Y)$.

Unbiased Measurement

We shall broaden our view to include ‘measurements’ that do not transfer the entire probability distribution from X to Y , but only the average.

Definition 11 (Unbiased Measurement) *Let $X \in \mathcal{A}$, $Y \in \mathcal{B}$ be Hermitean. An unbiased measurement \mathbf{M} of X with pointer Y is by definition a completely positive map $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ such that*

$$\mathbf{M}^*(\rho)(Y) = \rho(X) \quad \forall \rho \in \mathcal{S}(\mathcal{A}) \quad \text{or equivalently} \quad \mathbf{M}(Y) = X.$$

Equivalence is easily established with [K&R, p. 257]. Observing Y in state $\mathbf{M}^*(\rho)$ results in the same *average* as observing X in state ρ . But the probability distributions need not be the same.

Take *any* operation $\mathbf{T} : \mathcal{B} \rightarrow \mathcal{A}$. Take *any* $Y \in \mathcal{B}$. Then \mathbf{T} is automatically a measurement of $\mathbf{T}(Y)$ with pointer Y . Unbiased measurements are not hard to find.

Quality of Unbiased Measurement

Next, we define the quality σ of this unbiased measurement:

Definition 12 (Quality) Let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be an unbiased⁶ measurement of X with pointer Y . Then its quality σ is defined by

$$\sigma^2 \stackrel{\text{def}}{=} \sup\{ \text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_{\rho}(X) \mid \rho \in \mathcal{S}(\mathcal{A}) \}.$$

So σ tells us how much the uncertainty of the measurement result maximally exceeds the unavoidable amount of uncertainty inherent in the state ρ . In particular, when measuring an eigenstate of X , the variance in the measurement outcome will be less than or equal to σ^2 . It is intuitively clear (and we will prove shortly) that $\sigma^2 \geq 0$: the uncertainty of ρ in X is inherent and can never be diminished by some clever choice of measurement.

Lemma 13 The quality σ of an unbiased measurement $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ with Hermitean pointer $Y \in \mathcal{B}$ is $\|Y\|_{\mathbf{M}}$.

Proof:

If \mathbf{M} is a measurement of X , then X equals $\mathbf{M}(Y)$. This is apparent from $\rho(\mathbf{M}(Y) - X) = 0$ for all $\rho \in \mathcal{S}(\mathcal{A})$. Now for all $\rho \in \mathcal{S}(\mathcal{A})$

$$\begin{aligned} \text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_{\rho}(\mathbf{M}(Y)) &= \\ &= \left(\mathbf{M}^*(\rho)(Y^2) - (\mathbf{M}^*(\rho))(Y)^2 \right) - \left(\rho(\mathbf{M}(Y)^2) - \rho(\mathbf{M}(Y))^2 \right) \\ &= \rho(\mathbf{M}(Y^2)) - \rho(\mathbf{M}(Y)^2) \\ &= \rho(\mathbf{F}(Y, Y)) \end{aligned}$$

so that

$$\sigma^2 = \sup\{ \rho(\mathbf{F}(Y, Y)) \mid \rho \in \mathcal{S}(\mathcal{A}) \} = \|Y\|_{\mathbf{M}}^2.$$

This proves the assertion, as well as the positivity of σ^2 .

q. e. d.

From this, it is clear that if a measurement is perfect (in the sense of definition (10)), then it has optimal quality: $\sigma = 0$. One need only apply proposition (12) with $f(x) = x^2$.

It is time to look at some examples of unbiased measurement.

⁶In order to claim complete generality, we could abandon the demand that a measurement be unbiased: we would then introduce a *maximal bias* $\epsilon = \sup\{ |\mathbf{M}^*(\rho)(Y) - \rho(X)| \mid \rho \in \mathcal{S}(\mathcal{A}) \} = \|\mathbf{M}(Y) - X\|$. This would make *any* completely positive operation $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ a measurement of *any* $X \in \mathcal{A}$ with *any* pointer $Y \in \mathcal{B}$ and with a certain quality σ and maximal bias ϵ . The interested reader may consider it a home exercise to adapt the estimates to come, adding ϵ 's along the way. Good luck.

3.3.2 Examples of Unbiased Measurement

1. A direct⁷ observation of $X \in \mathcal{A}$, denoted $\mathbf{M} : \mathcal{C}(\mathbf{Spec}(X)) \rightarrow \mathcal{A}$ is defined by $f \mapsto f(X)$. In the dual (Schrödinger) picture, \mathbf{M}^* maps $\mathcal{S}(\mathcal{A})$ to $\mathcal{S}(\mathcal{C}(\mathbf{Spec}(X)))$, the probability distributions on $\mathbf{Spec}(X)$. It is completely positive. According to proposition (1), $\mathbf{M}^*(\rho) = \mathbb{P}_{\rho, X}$. If \mathbf{r} in $\mathcal{C}(\mathbf{Spec}(X))$ is the random variable $\mathbf{r}(\lambda) = \lambda$, then \mathbf{M} is a perfect measurement of X with pointer \mathbf{r} . Its quality is therefore $\sigma = 0$.
2. Indirect observation is also measurement: If $\mathbf{M}^* : \mathcal{S}(\mathcal{A}) \rightarrow \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ is defined by $\mathbf{M}^*(\rho) = \alpha^*(\rho \otimes \tau)$ for some automorphism α of $\mathcal{A} \otimes \mathcal{B}$ and for some $\tau \in \mathcal{S}(\mathcal{B})$, then it is completely positive. If $\mathbf{M}^*(\rho)(Y) = \rho(X)$ for all $\rho \in \mathcal{S}(\mathcal{A})$, then it is an unbiased measurement of X with pointer Y . It is perfect in the sense of definition (10) if and only if it is perfect in the sense of page 11.
3. Each automorphism is completely positive. If α is an automorphism of \mathcal{D} such that $\alpha(Y) = X$, then it is a perfect measurement of X with pointer Y . Its quality is automatically $\sigma = 0$.
4. Let $U \subset \mathbb{R}$ be compact in the Euclidean topology, and let \mathcal{A} be some von Neumann algebra. A ‘Positive Operator Valued Measure’ (POVM) (see [Hol, p. 51]) is defined⁸ as a mapping M from the Borel-measurable subsets of U into \mathcal{A} satisfying:

- $M(\emptyset) = 0$, $M(U) = \mathbb{I}$
- $M(V) \geq 0$ as an operator inequality for all measurable $V \subset U$.
- Suppose $\{V_j \mid j \in J\}$ is a countable decomposition of V , then $M(V) = \sum_{j \in J} M(V_j)$, where the sum converges in the weak sense.

By integrating bounded measurable functions $\mathcal{L}^\infty(U)$ on U over the POVM, M may be extended to a unital, positive operation $\mathbf{M} : \mathcal{L}^\infty(U) \rightarrow \mathcal{A}$. In short: $\mathbf{M}(f) \stackrel{\text{def}}{=} \int f(x)M(dx)$. It is completely positive due to the commutativity of $\mathcal{L}^\infty(U)$. Thus, we have an unbiased measurement of $\int xM(dx)$ with pointer $f : x \mapsto x$ and quality $\sigma^2 = \|\int x^2M(dx) - (\int xM(dx))^2\|$. The POVM is projection-valued iff $\sigma = 0$. It then reduces to the direct observation of example (1).

5. Davies (see [Dav, ch. 3]) also adopts the POVM as measurement. He gives a particularly nice example of an unbiased position measurement: Let $\mathcal{H} = L^2(\mathbb{R})$, and $\mathcal{A} = \mathcal{B}(\mathcal{H})$. Then the position observable X is defined by $(X\psi)(x) = x\psi(x)$. X has spectral measure $V \mapsto \mathbf{P}(V)$, defined by $(\mathbf{P}(V)\psi)(x) = \mathbb{I}_V(x)\psi(x)$.

Let f be a probability density on \mathbb{R} with zero mean. For Borel-sets $V \subset \mathbb{R}$, $M(V)$ is defined⁹ by $M(V) = \int_{-\infty}^{\infty} (f * \mathbb{I}_V)(x)\mathbf{P}(dx)$. f has the effect of blurring the outcome. In the limiting case that f is the Dirac δ -distribution on zero, $M(V) = \mathbf{P}(V)$.

M turns out to be a measurement of the position observable X indeed:

$\int xM(dx) = X$. Furthermore, Davies shows that the quality of M is exactly the r.m.s. value of f : $\sigma^2 = \mathbf{Var}(f)$.

⁷This kind of measurement is often called ‘von Neumann measurement’. We will reserve this designation for example 6 instead.

⁸In [Dav] and [Hol] even for non-compact U .

⁹ $f * g$ is the convolution of f and g . Explicitly, $(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$.

6. Suppose that the observable being measured has discrete spectrum, $X = \sum_i x_i \mathbf{P}_i$. Then there exists a so-called von Neumann measurement $\mathbf{N} : \mathcal{C}(\mathbf{Spec}(X)) \otimes \mathcal{A} \rightarrow \mathcal{A}$. It is defined by $f \otimes A \mapsto \sum_i f(i) \mathbf{P}_i A \mathbf{P}_i$. Constrained to $\mathcal{C}(\mathbf{Spec}(X)) \otimes \mathbb{I}$, it reduces again to direct observation. But \mathbf{N} also gives information about how the system is left behind.

\mathbf{N} is a perfect measurement of X with pointer $\mathfrak{x} \otimes \mathbb{I}$ (where $\mathfrak{x}(i) = x_i$) and quality $\sigma = 0$. The state $\mathbf{N}^*(\rho)|_{\mathcal{A} \otimes \mathbb{I}}$ is exactly the *collapsed state* of ρ after X -measurement: $\mathbf{N}^*(\rho)(A \otimes \mathbb{I}) = \rho(\sum_i \mathbf{P}_i A \mathbf{P}_i) = \mathbf{C}^*(\rho)(A)$.

Note that if $[A, X] = 0$ for some $A \in \mathcal{A}$, then $A = \sum_i \mathbf{P}_i A \mathbf{P}_i$ since it also commutes with the spectral projections. Consequently, $\mathbf{N}(A \otimes \mathbb{I}) = A$ and $\mathbf{N}^*(\rho)(A \otimes \mathbb{I}) = \rho(A)$.

7. Let $\mathbf{M} : M_2 \otimes C_2 \rightarrow M_2$ (C_2 are the 2×2 -diagonal matrices) be defined by

$$\mathbf{M}(A \otimes D) = \sum_{i=0,1} d_{ii} X_i^\dagger A X_i$$

with the matrices ($0 \leq \epsilon \leq 1/2$)

$$X_0 = \begin{pmatrix} \sqrt{1-\epsilon} & 0 \\ 0 & \sqrt{\epsilon} \end{pmatrix} \quad X_1 = \begin{pmatrix} \sqrt{\epsilon} & 0 \\ 0 & \sqrt{1-\epsilon} \end{pmatrix}$$

$M_2 \otimes C_2$ is isomorphic to $M_2 \oplus M_2$. There \mathbf{M} reads

$$\mathbf{M}(A \oplus B) = X_0^\dagger A X_0 + X_1^\dagger B X_1$$

From which one can verify complete positivity of \mathbf{M} . It is an unbiased measurement of σ_z with pointer $(1 - 2\epsilon)^{-1} \mathbb{I} \otimes \text{diag}(1, -1)$ and quality $\sigma = \frac{2\sqrt{\epsilon(1-\epsilon)}}{1-2\epsilon}$. For $\epsilon = 0$, \mathbf{M} reduces to the (perfect) von Neumann measurement of example 6. For $\epsilon = 1/2$, it has become completely useless: It produces a random outcome on C_2 , unrelated to the measured object. We'll come back to this example after proposition (19).

8. There are also silly examples of measurement: The identity is a perfect ($\sigma = 0$) measurement of any observable X with pointer X .
9. Any completely positive operation is a perfect ($\sigma = 0$) measurement of \mathbb{I} with pointer \mathbb{I} .

Definition (11) even seems to admit operations one would not call measurement. In spite of its generality though, much can be said about unbiased measurement and its quality. We shall give some examples of this, leaning heavily on lemmas (13) and (11).

3.3.3 Structure of a Perfect Measurement

But before that, we will exploit the fact that any completely positive operation \mathbf{M} acts as a homomorphism on the elements of \mathbf{M} -norm 0. For any Hermitean H we will denote by $\mathcal{C}(H)$ the C^* -algebra generated by \mathbb{I} and H .

Lemma 14 *Let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be a 4-positive operation, let $B \in \mathcal{B}$ be Hermitean. Among*

1. $\|B\|_{\mathbf{M}} = 0$.
2. \mathbf{M} is an isomorphism $\mathcal{C}(B) \rightarrow \mathcal{C}(\mathbf{M}(B))$.
3. $\mathbf{Spec}(B) = \mathbf{Spec}(\mathbf{M}(B))$ and $\mathbf{M}(f(B)) = f(\mathbf{M}(B))$ for all $f \in \mathcal{C}(\mathbf{Spec}(B))$.
4. $\|f(B)\|_{\mathbf{M}} = 0$ for all $f \in \mathcal{C}(\mathbf{Spec}(B))$.
5. \mathbf{M} maps the relative commutant B' into $\mathbf{M}(B)'$.

The following relations hold:

$$(1) \iff (2) \iff (3) \iff (4) \implies (5)$$

Proof:

(1) \Rightarrow (2): $\mathcal{C}(B)$ is the norm-closure in \mathcal{B} of the collection of polynomials in B . Similarly, $\mathcal{C}(\mathbf{M}(B))$ is the norm-closure in \mathcal{A} of the polynomials in $\mathbf{M}(B)$. By the multiplication theorem (11.3), we see that $\mathbf{M}(B^n) = \mathbf{M}(B)^n$. From this and linearity, one verifies that $\mathbf{M}(p(B)) = p(\mathbf{M}(B))$ and $\|p(B)\|_{\mathbf{M}} = 0$ for all polynomials p .

Now let $Q \in \mathcal{C}(B)$. By the Weierstrass theorem, there exist polynomials p_n such that $p_n(B) \rightarrow Q$ in norm. Since \mathbf{M} is automatically norm-continuous and since it maps $\mathcal{C}(B)$ densely into $\mathcal{C}(\mathbf{M}(B))$, we can verify that $\mathbf{M}(Q) \in \mathcal{C}(\mathbf{M}(B))$:

$$\mathbf{M}(Q) = \lim_{n \rightarrow \infty} \mathbf{M}(p_n(B)) = \lim_{n \rightarrow \infty} p_n(\mathbf{M}(B)) \in \mathcal{C}(\mathbf{M}(B)).$$

Similarly, $\|Q\|_{\mathbf{M}} = 0$ since

$$\begin{aligned} \mathbf{M}(Q^2) &= \mathbf{M}\left(\lim_{n \rightarrow \infty} p_n^2(B)\right) = \lim_{n \rightarrow \infty} \mathbf{M}(p_n^2(B)) = \lim_{n \rightarrow \infty} p_n^2(\mathbf{M}(B)) \\ &= \left(\lim_{n \rightarrow \infty} p_n(\mathbf{M}(B))\right)^2 = (\mathbf{M}(Q))^2. \end{aligned}$$

This means that the restriction of \mathbf{M} to $\mathcal{C}(B)$ is a C^* -isomorphism: if $Q \in \mathcal{C}(B)$, then even $\mathbf{M}(QA) = \mathbf{M}(Q)\mathbf{M}(A)$ for any $A \in \mathcal{B}$ by the multiplication theorem (11.3). Its image is therefore automatically norm-closed (see [K&R, p. 242]), and thus equal to $\mathcal{C}(\mathbf{M}(B))$.

(2) \Rightarrow (3): By the canonical isomorphism $f \mapsto f(B)$ known as the Gel'fand transform, $\mathcal{C}(B)$ is isomorphic to $\mathcal{C}(\mathbf{Spec}(B))$, the C^* -algebra of continuous functions on $\mathbf{Spec}(B)$ equipped with the supremum norm (see [K&R, p. 271]). Similarly, $\mathcal{C}(\mathbf{M}(B)) \sim \mathcal{C}(\mathbf{Spec}(\mathbf{M}(B)))$. \mathbf{M} acts as a continuous isomorphism mapping polynomials p on $\mathbf{Spec}(B)$ to the same p on $\mathbf{Spec}(\mathbf{M}(B))$ since $\mathbf{M}(p(B)) = p(\mathbf{M}(B))$. By continuity of $\mathbf{M}|_{\mathcal{C}(B)}$ and $\mathbf{M}^{-1}|_{\mathcal{C}(\mathbf{M}(B))}$, polynomials converge on $\mathbf{Spec}(B)$ if and only if they do on $\mathbf{Spec}(\mathbf{M}(B))$. Since the spectra are closed they must be the same, and by continuity of \mathbf{M} and the theorem of Stone-Weierstrass (the polynomials form a norm-dense set in the space of continuous functions), we now see that each continuous function is mapped to itself.

(3) \Rightarrow (4) : Let $f \in \mathcal{C}(\text{Spec}(B))$. Let $g(x) = f(x)^2$. Then by (3) we have $\mathbf{M}(f(B)^2) = \mathbf{M}(g(B)) = g(\mathbf{M}(B)) = f(\mathbf{M}(B))^2 = \mathbf{M}(f(B))^2$.

(4) \Rightarrow (1) : Trivial: choose $f(x) = x$.

(1) \Rightarrow (5) : Suppose that $A \in B'$, i. e. $[A, B] = 0$. Then by the multiplication theorem (11.3),

$$[\mathbf{M}(B), \mathbf{M}(A)] = \mathbf{M}([A, B]) - [\mathbf{M}(A), \mathbf{M}(B)] = 2i\mathbf{F}_{\mathbf{M}}(A, B) = 0.$$

q.e.d.

It is clear that perfect measurement (in the sense of definition (10)) satisfies $\sigma = 0$. But combining lemma (14) with proposition (12), we also obtain the converse:

Corollary 14.1 *Unbiased measurement (in the sense of definition (11)) is perfect (in the sense of definition (10)) if and only if it has quality $\sigma = 0$.*

Which is the moral obligation of any definition of quality. In the particular case of von Neumann algebras:

Corollary 14.2 *Let \mathcal{A} and \mathcal{B} be von Neumann algebras, and $A \in \mathcal{A}$, $B \in \mathcal{B}$ Hermitean. Suppose $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ measures X with pointer Y and quality $\sigma = 0$. Then \mathbf{M} is also a perfect measurement of all spectral projections $\mathbf{P}(V)$ of X , with pointer $\mathbf{Q}(V)$, the corresponding spectral projection of Y .*

Proof:

For any Borel set V and for any $\rho \in \mathcal{S}(\mathcal{A})$, we have seen that $\mathbb{P}_{\rho, X}(V) = \mathbb{P}_{\mathbf{M}^*(\rho), Y}(V)$. In particular, for all normal states ρ , this means that $\rho(\mathbf{P}(V)) = \mathbf{M}^*(\rho)(\mathbf{Q}(V))$, or $\rho(\mathbf{P}(V) - \mathbf{M}(\mathbf{Q}(V))) = 0$ for all normal states ρ . Therefore $\mathbf{M}(\mathbf{Q}(V)) = \mathbf{P}(V)$. Automatically, $\|\mathbf{Q}(V)\|_{\mathbf{M}} = 0$, since $\mathbf{M}(\mathbf{Q}^2(V)) = \mathbf{M}(\mathbf{Q}(V)) = \mathbf{P}(V) = \mathbf{P}^2(V)$.

q.e.d.

I hope that the paragraph above has given some credibility to our definitions of measurement and quality. They will form the basis of the rest of this thesis.

3.3.4 Simultaneous Measurement

As an appetizer, we'll use the C*-Cauchy-Schwarz inequality to generalize a well-known theorem. Perfect simultaneous measurement (i.e. measurement of two observables using two commuting pointers) can only be performed on commuting observables. The mathematical formulation below is based on Werner (see [Wer]), but the physical statement was known long before, see e.g. [Neu].

Proposition 15 (Joint Measurement) *Let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be a perfect ($\sigma_Y = \|Y\|_{\mathbf{M}} = 0$, $\sigma_{\tilde{Y}} = \|\tilde{Y}\|_{\mathbf{M}} = 0$) measurement of both $X \in \mathcal{A}$ and $\tilde{X} \in \mathcal{A}$ with commuting pointers $Y, \tilde{Y} \in \mathcal{B}$ respectively. (All Hermitean.) Then*

$$[X, \tilde{X}] = 0.$$

For example, let \mathbf{M}^* be an affine map from $\mathcal{S}(\mathcal{A})$ to the space of probability distributions on $\mathbf{Spec}(X) \times \mathbf{Spec}(\tilde{X})$ such that $\mathbb{P}_{\rho, X}$ and $\mathbb{P}_{\rho, \tilde{X}}$ are the marginal probability distributions of $\mathbf{M}^*(\rho)$. By the proof of proposition (1), the space of probability distributions on a probability space Ω can be identified with $\mathcal{S}(\mathcal{C}(\Omega))$, the state-space of the C^* -algebra $\mathcal{C}(\Omega)$. Due to the abelianness of $\mathcal{C}(\Omega)$, \mathbf{M}^* must be the dual of a completely positive map. It is therefore a joint measurement in the sense of proposition (15). We see that the kind of mapping constructed in proposition (2), vital to the interpretation of quantum mechanics, simply does not exist if $[X, \tilde{X}] \neq 0$.

We'll prove proposition (15) along with a Heisenberg relation-like generalization¹⁰, relating the product of both measurement qualities with the lack of commutativity.

Proposition 16 (Generalized Joint Measurement) *Let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be an unbiased measurement of $X \in \mathcal{A}$ and $\tilde{X} \in \mathcal{A}$, both Hermitean, with commuting Hermitean pointers $Y, \tilde{Y} \in \mathcal{B}$ respectively. Then for the qualities $\sigma_Y = \|Y\|_{\mathbf{M}}$ and $\sigma_{\tilde{Y}} = \|\tilde{Y}\|_{\mathbf{M}}$ the following relation holds:*

$$2\sigma_Y\sigma_{\tilde{Y}} \geq \|[X, \tilde{X}]\|.$$

Proof:

$$\begin{aligned} \|[X, \tilde{X}]\| &= \|\mathbf{M}([Y, \tilde{Y}]) - [\mathbf{M}(Y), \mathbf{M}(\tilde{Y})]\| = \|2\Im(\mathbf{F}(Y, \tilde{Y}))\| \leq 2\|Y\|_{\mathbf{M}}\|\tilde{Y}\|_{\mathbf{M}} = \\ &2\sigma_Y\sigma_{\tilde{Y}}, \text{ proving both propositions (15) and (16).} \end{aligned}$$

q. e. d.

¹⁰[Hol, p. 90] already gives a generalization for POVM's. However, this involves only $\mathbf{Var}_{\mathbf{M}^*(\rho)}(Y)$ instead of $\mathbf{Var}_{\mathbf{M}^*(\rho)}(Y) - \mathbf{Var}_{\rho}(X)$, thus staying much closer to the Heisenberg uncertainty relations.

3.4 The Heisenberg Principle

The so-called Heisenberg principle¹¹ may be formulated as follows:

When an outside observer extracts quantum-information from a system, it is impossible to leave all states unaltered.

As for a mathematical formulation and proof, due to R. Werner (see [Wer]):

Proposition 17 (Heisenberg Principle) *Let $\mathbf{M} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$ be an unbiased measurement of any Hermitean $X \in \mathcal{A}$ with any Hermitean pointer $\mathbb{I} \otimes Y \in \mathbb{I} \otimes \mathcal{B}$. Suppose that \mathbf{M} leaves states on \mathcal{A} undisturbed: $\mathbf{M}^*(\rho)(A \otimes \mathbb{I}) = \rho(A) \quad \forall A \in \mathcal{A} \quad \forall \rho \in \mathcal{S}(\mathcal{A})$. Then X is in the centre of \mathcal{A} .*

Proof:

In circumstances above, $\mathbf{M}(A \otimes \mathbb{I}) = A$ for all $A \in \mathcal{A}$. Therefore also $\mathbf{M}(A^\dagger A \otimes \mathbb{I}) = A^\dagger A$ for all $A \in \mathcal{A}$. Thus $\|A \otimes \mathbb{I}\|_{\mathbf{M}}^2 = 0$, which entails $[\mathbf{M}(A \otimes \mathbb{I}), \mathbf{M}(\mathbb{I} \otimes Y)] = 0$, i.e. $[A, X] = 0$ for all A in \mathcal{A} .

q.e.d.

So the only information (of any quality) that can be obtained without disturbing the original system is information about central elements.

We have good reason to consider ‘central information’ as ‘classical information’: In the fully classical case, the algebra \mathcal{A} is abelian. All observables are central, so all information is freely accessible. In the archetypal quantum case however, the algebra \mathcal{A} in question is $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on some Hilbert space \mathcal{H} . In this case, \mathbb{I} is the only central element (modulo \mathbb{C}), so that no information can be gained without disturbing the system.

3.4.1 Global Generalization

So we have grounds to examine the norm distance of X to the centre, $d(X, \mathcal{Z}) = \inf\{\|X - Z\| \mid Z \in \mathcal{Z}\}$. One may think of $d(X, \mathcal{Z})$ as quantifying the amount of ‘quantumness’ in X , as it determines the maximal amount of non-commutativity with X in the sense below:

Lemma 18 *Let X be a Hermitean element of a finite-dimensional von Neumann algebra \mathcal{A} . Then $d(X, \mathcal{Z})$ is the smallest number c such that $\|[A, X]\| \leq 2c\|A\|$ for all $A \in \mathcal{A}$.*

Proof:

Finite or infinite dimensional algebra, it is clear that for any $A \in \mathcal{A}$, $\|[A, X]\| = \|[A, X - Z]\| \leq 2\|A\|\|X - Z\|$ for any $Z \in \mathcal{Z}(\mathcal{A})$. Taking the infimum over Z , we obtain $\|[A, X]\| \leq 2d(X, \mathcal{Z})\|A\|$. This means that we are finished if we find $A \in \mathcal{A}$, $A \neq 0$ such that $\|[A, X]\| = 2d(X, \mathcal{Z})\|A\|$.

¹¹Not to be confused with the Heisenberg uncertainty relations in corollary (11.2).

Now for any von Neumann algebra \mathcal{A} , there exists a so-called decomposition over the centre (see [K&R, ch. 14]). For finite-dimensional von Neumann algebras, this simply means that \mathcal{A} is isomorphic to $M_{n_1} \oplus \dots \oplus M_{n_k}$ for some $k; n_1, \dots, n_k \in \mathbb{N}$, where M_{n_i} is the algebra of $n_i \times n_i$ matrices. So $X = X_1 \oplus \dots \oplus X_k$ with $X_k \in M_{n_k}$. Each X_k can be brought in diagonal form by some unitary transformation U_k . So by the isomorphism $U_1 \oplus \dots \oplus U_k$, we may think of X as

$$\text{diag}(\lambda_{(1,1)}, \dots, \lambda_{(1,n_1)}) \oplus \dots \oplus \text{diag}(\lambda_{(k,1)}, \dots, \lambda_{(k,n_k)})$$

with $\lambda_{(i,1)} \geq \dots \geq \lambda_{(i,n_i)}$ the eigenvalues of X_i in decreasing order. Let $r_i \stackrel{\text{def}}{=} \frac{1}{2} (\lambda_{(i,1)} - \lambda_{(i,n_i)})$ be the spectral radius of X_i . Let $t_i \stackrel{\text{def}}{=} \frac{1}{2} (\lambda_{(i,1)} + \lambda_{(i,n_i)})$. Let

$$\begin{aligned} \tilde{X} &\stackrel{\text{def}}{=} X - (t_1 \mathbb{I} \oplus \dots \oplus t_k \mathbb{I}) \\ &= \text{diag}(\tilde{\lambda}_{(1,1)}, \dots, \tilde{\lambda}_{(1,n_1)}) \oplus \dots \oplus \text{diag}(\tilde{\lambda}_{(k,1)}, \dots, \tilde{\lambda}_{(k,n_k)}) \end{aligned}$$

with $\tilde{\lambda}_{(i,1)} = r_i$, $\tilde{\lambda}_{(i,n_i)} = -r_i$. \tilde{X} and X differ only by the central element $t_1 \mathbb{I} \oplus \dots \oplus t_k \mathbb{I} \in \mathcal{Z}$. Furthermore, $\|\tilde{X}\| = \max_i r_i$. Take that maximal r_i and isolate the highest and lowest eigenvectors

$$0 \oplus \dots \oplus 0 \oplus \psi_{+,-} \oplus 0 \oplus \dots \oplus 0$$

in $\mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_k}$. Construct

$$A = 0 \oplus \dots \oplus 0 \oplus (|\psi_+\rangle\langle\psi_-| + |\psi_-\rangle\langle\psi_+|) \oplus 0 \oplus \dots \oplus 0$$

in \mathcal{A} . Now, since $X - \tilde{X} \in \mathcal{Z}$, we see that

$$[X, A] = [\tilde{X}, A] = 0 \oplus \dots \oplus 0 \oplus 2r_i(|\psi_+\rangle\langle\psi_-| - |\psi_-\rangle\langle\psi_+|) \oplus 0 \oplus \dots \oplus 0.$$

So $\|[X, A]\| = \|[\tilde{X}, A]\| = 2r_i = 2\|\tilde{X}\|\|A\|$. But since already $d(X, \mathcal{Z}) \leq \|\tilde{X}\|$ and $\|[\tilde{X}, A]\| \leq 2d(X, \mathcal{Z})\|A\|$, we see that

$$2d(X, \mathcal{Z})\|A\| \leq 2\|\tilde{X}\|\|A\| = \|[X, A]\| \leq 2d(X, \mathcal{Z})\|A\|.$$

So $\|\tilde{X}\| = d(X, \mathcal{Z})$ and $\|[X, A]\| = 2d(X, \mathcal{Z})\|A\|$.

q.e.d.

We are now looking for generalizations of proposition (17) of the following form: suppose you allow some (small) disturbance of the states on the original algebra. How much information can be gained maximally? Note that in the proposition below, all that is used about the pointer is that it commutes with $\mathcal{A} \otimes \mathbb{I}$.

Proposition 19 (Generalized Heisenberg Principle) *Let \mathcal{A} be a finite-dimensional von Neumann algebra with centre \mathcal{Z} . Let \mathcal{B} be an arbitrary von Neumann algebra and let $Y \in \mathcal{B}$, $X \in \mathcal{A}$ Hermitean. Let \mathbf{M} be completely positive $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$ such that:*

- $\mathbf{M}(\mathbb{I} \otimes Y) = X$ and $\|\mathbb{I} \otimes Y\|_{\mathbf{M}} = \sigma$, i. e. \mathbf{M} is an unbiased measurement of X with pointer $\mathbb{I} \otimes Y$ and quality σ .
- $\|\mathbf{M}^*(\rho)|_{\mathcal{A} \otimes \mathbb{I}} - \rho\| \leq \Delta \quad \forall \rho \in \mathcal{S}(\mathcal{A})$ for some $0 < \Delta < 1$.

Then

$$\sigma \geq d(X, \mathcal{Z}) \frac{1 - \Delta}{\sqrt{3\Delta}}.$$

Proof:

We move to the Heisenberg picture: $\|\mathbf{M}(A \otimes \mathbb{I}) - A\| \leq \Delta\|A\|$ because $\rho(\mathbf{M}(A \otimes \mathbb{I}) - A) \leq \Delta\|A\|$ for all $\rho \in \mathcal{S}(\mathcal{A})$. For notational convenience, we introduce an operation $\mathbf{T} : \mathcal{A} \rightarrow \mathcal{A}$ defined by $\mathbf{T}(A) \stackrel{\text{def}}{=} \mathbf{M}(A \otimes \mathbb{I})$ and the map $\mathbf{D} : \mathcal{A} \rightarrow \mathcal{A}$ defined by $\mathbf{D}(A) \stackrel{\text{def}}{=} \mathbf{T}(A) - A$. The former is the effect of measurement on the measured system \mathcal{A} , the latter satisfies $\|\mathbf{D}(A)\| \leq \Delta\|A\|$. Since $\mathbf{D}(A)^\dagger \mathbf{D}(A) \geq 0$, we may estimate

$$\begin{aligned} \|A \otimes \mathbb{I}\|_{\mathbf{M}}^2 &= \|\mathbf{F}_{\mathbf{T}}(A, A)\| \\ &\leq \|\mathbf{F}_{\mathbf{T}}(A, A) + \mathbf{D}(A)^\dagger \mathbf{D}(A)\| \\ &= \|\mathbf{T}(A^\dagger A) - \mathbf{T}(A^\dagger) \mathbf{T}(A) + \mathbf{D}(A)^\dagger \mathbf{D}(A)\| \\ &= \|(\mathbf{D}(A^\dagger A) + A^\dagger A) - (\mathbf{D}(A) + A)^\dagger (\mathbf{D}(A) + A) + \mathbf{D}(A)^\dagger \mathbf{D}(A)\| \\ &= \|(\mathbf{D}(A^\dagger A) - \mathbf{D}(A)^\dagger A - A^\dagger \mathbf{D}(A))\| \\ &\leq 3\Delta\|A\|^2. \end{aligned}$$

Since $\Delta < 1$, \mathbf{T} must be injective because it is linear and because

$$\mathbf{T}(A) = 0 \implies \|A\| = \|\mathbf{D}(A)\| \leq \Delta\|A\| \implies \|A\| = 0.$$

Since \mathcal{A} is finite-dimensional, this implies that \mathbf{T} is onto. Furthermore, $\|\mathbf{T}(A) - A\| \leq \Delta\|A\|$ implies $\|\mathbf{T}(A)\| - \|A\| \leq \Delta\|A\|$ and hence $\|A\| \leq \|\mathbf{T}(A)\|/(1 - \Delta)$. By the C*-Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} \|[X, \mathbf{T}(A)]\| &= \|[\mathbf{M}(\mathbb{I} \otimes Y), \mathbf{M}(A \otimes \mathbb{I})]\| \\ &\leq 2\|(\mathbb{I} \otimes Y)\|_{\mathbf{M}}\|(A \otimes \mathbb{I})\|_{\mathbf{M}} \\ &\leq 2\sigma\sqrt{3\Delta}\|A\| \\ &\leq 2\sigma\frac{\sqrt{3\Delta}}{1 - \Delta}\|\mathbf{T}(A)\|. \end{aligned}$$

So, since \mathbf{T} is onto, $\sigma\frac{\sqrt{3\Delta}}{1 - \Delta}$ is a number c such that $\|[X, A]\| \leq 2c\|A\| \quad \forall A \in \mathcal{A}$. By lemma (18), $d(X, \mathcal{Z})$ is the smallest such number. Thus $d(X, \mathcal{Z}) \leq \sigma\frac{\sqrt{3\Delta}}{1 - \Delta}$.

q.e.d.

If $\Delta = 0$, the last line of the proof reduces to proposition (17) for finite dimensional algebras: unbiased measurement is only possible if X is central.

If $\Delta \neq 0$, proposition (19) says that $\sigma \geq d(X, \mathcal{Z})^{\frac{1-\Delta}{\sqrt{3\Delta}}}$: given a non-central $X \in \mathcal{A}$ to be measured with maximal disturbance Δ . Then the attainable measurement quality σ is worse than $d(X, \mathcal{Z})^{\frac{1-\Delta}{\sqrt{3\Delta}}}$. We see that σ becomes deplorable if Δ is lowered to zero.

For example, let's look again at the unbiased measurement $\mathbf{M} : M_2 \otimes C_2 \rightarrow M_2$, discussed in example 7 on page 45. Explicit calculation shows that \mathbf{M} satisfies the conditions of proposition (19) for $\Delta = 1 - 2\sqrt{\epsilon(1-\epsilon)}$. It yields the estimate $\sigma \geq 2\sqrt{\frac{\epsilon(1-\epsilon)}{3-6\sqrt{\epsilon(1-\epsilon)}}}$

whereas the real quality of \mathbf{M} equals $\frac{2\sqrt{\epsilon(1-\epsilon)}}{1-2\epsilon}$. In this case (and probably in general) the estimate is rather crude¹². But it does contain some general features of the curve $\sigma(\epsilon)$, notably $\lim_{\epsilon \uparrow \frac{1}{2}} \sigma(\epsilon) = \infty$ and $\lim_{\epsilon \downarrow 0} \sigma(\epsilon) = 0$.

3.4.2 Local Generalization

We have extended the Heisenberg principle by demanding that all states on \mathcal{A} are perturbed in norm only slightly instead of not at all. Another way of 'extending' it is by demanding that all states are left exactly in place, but only with respect to some observables:

Proposition 20 (Generalized Heisenberg Principle) *Let $X \in \mathcal{A}$, $Y \in \mathcal{B}$ be Hermitian. Let $\mathbf{M} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$ be an unbiased measurement of X with pointer $\mathbb{I} \otimes Y$ and quality σ . Let $0 \neq A \in \mathcal{A}$ with $\|[X, A]\| = \delta\|A\|$ be such that*

$$\mathbf{M}^*(\rho)(A \otimes \mathbb{I}) = \rho(A) \quad \forall \rho \in \mathcal{S}(\mathcal{A}).$$

Then

$$\sigma \geq \delta/2.$$

Proof:

Of course $\mathbf{M}(A \otimes \mathbb{I}) = A$. By the C*-Cauchy-Schwarz inequality,

$$\begin{aligned} \delta\|A\| &= \|[X, A]\| = \|[\mathbf{M}(\mathbb{I} \otimes Y), \mathbf{M}(A \otimes \mathbb{I})] - \mathbf{M}([\mathbb{I} \otimes Y, A \otimes \mathbb{I}])\| \\ &= \|2i\Im \mathbf{F}_{\mathbf{M}}(\mathbb{I} \otimes Y, A \otimes \mathbb{I})\| \leq 2\|\mathbb{I} \otimes Y\|_{\mathbf{M}}\|A \otimes \mathbb{I}\|_{\mathbf{M}} \leq 2\sigma\|A\|. \end{aligned}$$

q.e.d.

For example, let $\mathcal{A} = \bigotimes_{i=1}^N M_2$. Let $\sigma_x, \sigma_y, \sigma_z$ be the Pauli spin-matrices in M_2 , and denote by σ_α^i the observable $\mathbb{I} \otimes \dots \otimes \mathbb{I} \otimes \sigma_\alpha \otimes \mathbb{I} \otimes \dots \otimes \mathbb{I}$. Let $X = \frac{1}{N} \sum_{i=1}^N \sigma_x^i$ and $A = \frac{1}{N} \sum_{i=1}^N \sigma_y^i$ be the average spin in the x - and y -directions. Then $\|[X, A]\| = \frac{2}{N}\|A\|$, so that any measurement of X leaving A untouched automatically has quality $\sigma \geq \frac{1}{N}$. Accurate average spin measurement in all directions simultaneously is only possible in large systems.

¹²Note that the hideous $\sqrt{3\Delta}$ comes from the estimate $\|A \otimes \mathbb{I}\|_{\mathbf{M}} \leq \sqrt{3\Delta}\|A\|$. This is the part of the proof where the crudeness comes in: In this particular example $\|A \otimes \mathbb{I}\|_{\mathbf{M}} \leq (1-2\epsilon)\|A\| \quad \forall A \in \mathcal{A}$. Taking the proof of proposition (19) from there would yield the true σ as an estimate.

3.5 State Reduction and Collapse

Until now we have only looked upon the Heisenberg principle from one side: given a certain amount of disturbance, how much information can one gain from a system? On the flip side, we may consider the following question. Given a measurement of a certain quality, how does this perturb the system? In case of a perfect measurement ($\sigma = \|Y\|_{\mathbf{M}} = 0$), lemma (14) and its corollaries give fairly detailed restrictions on the structure of \mathbf{M} . For example, \mathbf{M} has to map Y' into $\mathbf{M}(Y)'$. In the Schrödinger picture, this translates into a particularly nice answer to the above question known as ‘state collapse’.

3.5.1 State Reduction

Let ρ be a state on \mathcal{A} , and $X \in \mathcal{A}$. In definition (1), we have defined the state ρ_X on \mathcal{A} by

$$\rho_X(A) = \frac{\rho(X^\dagger A X)}{\rho(X^\dagger X)}.$$

Let \mathcal{A} and \mathcal{B} be von Neumann algebras, and let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be a measurement of $X \in \mathcal{A}$ with pointer $Y \in \mathcal{B}$ and quality $\sigma = 0$. Let $\mathbf{P}(V)$ be the spectral projections of X , $\mathbf{Q}(V)$ those of Y . In corollary (14.2), we have seen that \mathbf{M} measures perfectly the spectral projections of X with the corresponding ones of Y : $\mathbf{M}(\mathbf{Q}(V)) = \mathbf{P}(V)$ and $\|\mathbf{Q}(V)\|_{\mathbf{M}} = 0$. Under these circumstances, and if both ρ and \mathbf{M} are normal (i.e. weakly continuous), we have seen 2 examples of the above definition:

- We may look at the normal state $\mathbf{M}^*(\rho)$ resulting from measurement. An observer $\mathcal{C} \ni Y$ may condition all its observations on the pointer outcome V . On page 9, the conditioned probability is shown to be induced by the state $(\mathbf{M}^*(\rho))_{\mathbf{Q}(V)}$:

$$\mathbb{P}_{\mathbf{M}^*(\rho), B, Y}([B \text{ in } W] \mid [Y \text{ in } V]) = \mathbb{P}_{(\mathbf{M}^*(\rho))_{\mathbf{Q}(V)}, B}([B \text{ in } W]).$$

- If *any* perfect measurement of Y yields outcome in V , the system is experimentally known to be in state $\rho_{\mathbf{P}(V)}$ prior to measurement. This is called the *reduced state*.

We prove proposition (3) in the completely positive setting, showing that the commuting diagram on page 15 remains valid for all completely positive perfect measurements \mathbf{M} :

Proposition 21 (Reduction) *Let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be a measurement of $X \in \mathcal{A}$ with pointer $Y \in \mathcal{B}$ and quality $\sigma = 0$. Then for all states $\rho \in \mathcal{A}$:*

$$(\mathbf{M}^* \rho)_Y = \mathbf{M}^*(\rho_X).$$

Proof:

Writing out the definitions, we need to prove that for all $A \in \mathcal{A}$:

$$\rho(\mathbf{M}(Y^\dagger AY))\rho(Y^\dagger Y) = \rho(X^\dagger \mathbf{M}(A)X)\rho(\mathbf{M}(Y^\dagger Y)).$$

Since $\|Y\|_{\mathbf{M}} = 0$, $\mathbf{M}(Y^\dagger Y) = X^\dagger X$. By the multiplication theorem (11.3), $\mathbf{M}(Y^\dagger AY) = X^\dagger \mathbf{M}(A)X$. Letting ρ act on the above proves the assertion.

q. e. d.

Taking $\mathbf{Q}(V)$ for Y and $\mathbf{P}(V)$ for X in the above proposition, $(\mathbf{M}^*(\rho))_{\mathbf{Q}(V)} = \mathbf{M}^*(\rho_{\mathbf{P}(V)})$: reducing a measured state according to pointer outcome and measuring a reduced state amounts to the same thing. In particular, for $[B, Y] = 0$,

$$\mathbb{P}_{\mathbf{M}^*(\rho), B, Y}([B \text{ in } W] \mid [Y \text{ in } V]) = \mathbb{P}_{(\mathbf{M}^*(\rho_{\mathbf{P}(V)}), B}([B \text{ in } W]).$$

If you measure X with pointer Y and register an outcome in V , then direct observation of all $B \in \mathcal{C}$ will be as if, prior to measurement, the system had been in the reduced state $\rho_{\mathbf{P}(V)}$.

Of course this is also true for all indirect observations, as long as the outcome is not erased from the original pointer Y . If, after \mathbf{M}^* , an operation \mathbf{N}^* takes place, leaving the pointer Y and its projections untouched, then we may simply apply proposition (21) to $\mathbf{N}^* \circ \mathbf{M}^*$ instead of \mathbf{M}^* . This explains why state reduction $\rho \mapsto \rho_{\mathbf{P}(V)}$ is observed by \mathcal{C} as long as \mathcal{C} observes the outcome X in V indirectly.

3.5.2 State Collapse

Note the essential difference between conditioning on the outcome $[X \text{ in } V]$ and forming the X -reduced state: the former can only be done on X' , the latter on all of \mathcal{A} . Let X have spectral measure $V \mapsto \mathbf{P}(V)$. On X' , ρ is a harmless classical superposition of X -reduced states:

$$\rho(A) = \sum_I \rho(\mathbf{P}(V_i)A) = \sum_I \rho(\mathbf{P}(V_i)A\mathbf{P}(V_i)) = \sum_I \rho(\mathbf{P}(V_i))\rho_{\mathbf{P}(V_i)}(A) \quad \forall A \in X'.$$

As far as $A \in X'$ is concerned, a system in state ρ is simply in state $\rho_{\mathbf{P}(V_i)}$ with probability $\rho(\mathbf{P}(V_i))$. On page 8, we have introduced the *collapse operation* $\mathbf{C} : \mathcal{A} \rightarrow \mathcal{A}$, defined by $\mathbf{C}(A) \stackrel{\text{def}}{=} \sum_I \mathbf{P}(V_i)A\mathbf{P}(V_i)$. Then $\mathbf{C}^*(\rho)$ is known as the *collapsed* state: $\mathbf{C}^*(\rho) = \sum_I \rho(\mathbf{P}(V_i))\rho_{\mathbf{P}(V_i)}$.

When you remove the restriction $A \in X'$, ρ is *not* just the classical superposition of its reduced states. The difference between ρ and $\mathbf{C}^*(\rho)$ on \mathcal{A} is experimentally¹³ observable, e.g. by two-slit-experiments or along the lines of page 6.

In summary, a state ρ is only a classical superposition of the X -reduced states $\rho_{\mathbf{P}(V_i)}$ on X' , the commutant of X . But proposition (21) shows that after measurement the

¹³Of course, reduction on classical (central) observables is the exception to the rule. For a central observable Z , the relative commutant Z' equals \mathcal{A} , and any state may at any time be safely considered a classical superposition of Z -reduced states.

tables are turned: $\mathbf{M}^*(\rho)$ is a classical superposition of the $\mathbf{M}^*(\rho_{\mathbf{P}(V_i)})$ on Y' , not X' . In other words, $\mathbf{M}^*(\rho) = \mathbf{M}^* \circ \mathbf{C}^*(\rho)$ on Y' . This can also be seen directly from the structure of \mathbf{M} .

Proposition 22 (Collapse) *Let \mathcal{A}, \mathcal{B} be von Neumann algebras. Let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be a perfect ($\sigma = 0$) measurement of a Hermitean $X \in \mathcal{A}$ with pointer $Y \in \mathcal{B}$. Let $V \mapsto \mathbf{P}(V)$ be the spectral measure of X . Let $\{V_i \mid i \in I\}$ be a countable decomposition of $\mathbf{Spec}(X)$. Then a collapse operation $\mathbf{C} : \mathcal{A} \rightarrow \mathcal{A}$ is defined by $\mathbf{C}(A) \stackrel{\text{def}}{=} \sum_I \mathbf{P}(V_i) A \mathbf{P}(V_i)$. In this situation, we have for any $\rho \in \mathcal{S}(\mathcal{A})$:*

$$\mathbf{M}^*(\rho) = \mathbf{M}^* \circ \mathbf{C}^*(\rho) \quad \text{on } Y'.$$

Proof:

\mathbf{M} maps Y' into X' , and \mathbf{C} leaves X' pointwise fixed. Therefore, if $B \in Y'$, we have $\mathbf{M}^* \circ \mathbf{C}^*(\rho)(B) = \rho(\mathbf{C} \circ \mathbf{M}(B)) = \rho(\mathbf{M}(B)) = \mathbf{M}^*(\rho)(B)$.

q.e.d.

The above proposition states that when you measure X in state ρ , and then restrict attention to the commutant of the pointer, then the system will behave as if it had been in the collapsed state $\mathbf{C}^*(\rho)$ prior to measurement. It generalizes the diagram on page 15 to all completely positive perfect measurements.

A measurement of X with pointer Y is called *repeatable* if immediate repetition of the measurement would yield the same result, i.e. if $\mathbf{M}^*(\rho) = \rho$ on $\mathcal{C}(X)$. Then $\|X\|_{\mathbf{M}}$ must be¹⁴ 0, so according to the definitions \mathbf{M} is also a perfect measurement of X with pointer X . According to the above proposition then, $\mathbf{M}^*(\rho)$ equals $\mathbf{M}^* \circ \mathbf{C}^*(\rho)$ not only on Y' , but also on X' .

For example, if $\mathbf{M} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$ measures X with pointer $\mathbb{I} \otimes Y$ in a repeatable way, then the distinction between states that are collapsed or intact prior to measurement can, after measurement, neither be made by observables of the form $A \otimes \mathbb{I}$ nor of the form $\mathbb{I} \otimes B$.

3.5.3 Generalized State Collapse

Very well. For perfect ($\sigma = 0$) measurements of X with pointer Y , the essence of state-collapse is that \mathbf{M} maps Y' into X' . The way to think of X' is the following: in a finite-dimensional algebra, X can be decomposed into projections as $X = \sum_i \lambda_i \mathbf{P}_i$. And of course $\mathbf{M}(B) = \sum_{i,j} \mathbf{P}_i \mathbf{M}(B) \mathbf{P}_j$ for any $B \in Y'$. Now $\mathbf{M}(B) \in X'$ means $\mathbf{M}(B) = \sum_i \mathbf{P}_i \mathbf{M}(B) \mathbf{P}_i$; then $\mathbf{M}(B)$ contains only diagonal blocks, like

$$X = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{pmatrix} \implies \mathbf{M}(B) = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix}.$$

¹⁴Since $\mathbf{M}(f(X)) = f(X)$ for all continuous f on $\mathbf{Spec}(X)$, this is certainly true for $f(x) = x^2$, whence $\mathbf{F}(X, X) = 0$.

And of course this is just the disappearance of coherences between eigenstates of X with different eigenvalues. In this light, an approximate collapse proposition would have to be something that says how small the ‘off-diagonal blocks’ of $\mathbf{M}(B)$ get, provided that B commutes rather well with the pointer Y , and that the quality of measurement σ is not too bad. In other words: a Heisenberg-equivalent of lemma (7).

Proposition 23 (Generalized Collapse in the Heisenberg-Picture) *Let \mathcal{A} be a von Neumann algebra. Let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be an unbiased measurement of $X \in \mathcal{A}$ with pointer $Y \in \mathcal{B}$ (both Hermitean) and quality σ . Let $\mathbf{Spec}(X) \supseteq V \mapsto \mathbf{P}(V)$ denote the projection valued measure belonging to X . Suppose $B \in \mathcal{B}$ is a Hermitean element such that $\|[Y, B]\| = \delta\|B\|$. Then*

$$\|\mathbf{P}([x, x + \epsilon])\mathbf{M}(B)\mathbf{P}([y, y + \epsilon])\| \leq \frac{\delta + 2\sigma + \epsilon}{|x - y|} \|B\|.$$

Proof:

By the C*-Cauchy-Schwarz inequality,

$$\begin{aligned} \|[X, \mathbf{M}(B)]\| &= \|\mathbf{M}([Y, B]) + 2i\Im \mathbf{F}(B, Y)\| \\ &\leq \|[Y, B]\| + 2\|Y\|_{\mathbf{M}}\|B\|_{\mathbf{M}} \\ &\leq (\delta + 2\sigma)\|B\|. \end{aligned} \tag{3.3}$$

In order to dehorrrify our formulas, we introduce some notation. First of all, we define $u_n \stackrel{\text{def}}{=} u + n\epsilon$ for $u \in \mathbb{R}$, $n \in \mathbb{Z}/2$. Secondly, $\mathbf{P}_{u,n} \stackrel{\text{def}}{=} \mathbf{P}([u_n, u_{n+1}])$. And finally, we define $X_u = \sum_{n \in \mathbb{Z}} u_{n+\frac{1}{2}} \mathbf{P}_{u,n}$. This is an approximation of X by a step function operator, so that

$$\|X - X_u\| \leq \epsilon/2 \tag{3.4}$$

This leads us to

$$\begin{aligned} \|X_x \mathbf{M}(B) - \mathbf{M}(B) X_y\| &= \\ &= \|[X, \mathbf{M}(B)] + \mathbf{M}(B)(X - X_y) - (X - X_x)\mathbf{M}(B)\| \\ &\leq (\delta + 2\sigma)\|B\| + \epsilon\|\mathbf{M}(B)\| \\ &\leq (\delta + \epsilon + 2\sigma)\|B\|. \end{aligned} \tag{3.5}$$

Now for any 2 projections \mathbf{P} and \mathbf{Q} and for any A in \mathcal{A} , we have $\|\mathbf{P}A\mathbf{Q}\| \leq \|\mathbf{P}\|\|A\|\|\mathbf{Q}\| \leq \|A\|$. We use this in the second step below. In the first step, we use that $\mathbf{P}_{u,n}\mathbf{P}_{u,0} = \delta_{n,0}\mathbf{P}_{u,0}$ for $n \in \mathbb{N}$. And in the third step, we make use of $\sum_{n \in \mathbb{Z}} \mathbf{P}_{u,n} = \mathbb{I}$. We then obtain

$$\begin{aligned} |x - y| \|\mathbf{P}([x, x + \epsilon])\mathbf{M}(B)\mathbf{P}([y, y + \epsilon])\| &= \\ &= \left\| \mathbf{P}_{x,0} \left(\sum_{m,n \in \mathbb{Z}} (x_{n+\frac{1}{2}} - y_{m+\frac{1}{2}}) \mathbf{P}_{x,n} \mathbf{M}(B) \mathbf{P}_{y,m} \right) \mathbf{P}_{y,0} \right\| \\ &\leq \left\| \sum_{m,n \in \mathbb{Z}} (x_{n+\frac{1}{2}} - y_{m+\frac{1}{2}}) \mathbf{P}_{x,n} \mathbf{M}(B) \mathbf{P}_{y,m} \right\| \end{aligned}$$

$$\begin{aligned}
&= \left\| \sum_{n \in \mathbb{Z}} x_{n+\frac{1}{2}} \mathbf{P}_{x,n} \mathbf{M}(B) - \sum_{m \in \mathbb{Z}} y_{m+\frac{1}{2}} \mathbf{M}(B) \mathbf{P}_{y,m} \right\| \\
&= \|X_x \mathbf{M}(B) - X_y \mathbf{M}(B)\|.
\end{aligned} \tag{3.6}$$

With inequality (3.5), we now have what we wanted.

q.e.d.

Even if $\delta = 0$, the norm distance between $\mathbf{M}(B)$ and X' does not go to zero as $\sigma \downarrow 0$. Suppose that X has a continuous spectrum. If $\sigma \neq 0$, however small, we can always choose x and y in $\mathbf{Spec}(X)$ so that $|x - y| \leq 2\sigma$. The above proposition then becomes trivial. It allows for large off-diagonal elements as long as they are close to the diagonal.

This is physically relevant: suppose that the internal energy X of a block of iron is measured with an accuracy σ of few microjoules. Clearly this measurement does not produce decoherence between energy-states inside the atoms, i.e. eigenstates with energies x and y differing several eV. Indeed, the estimates ‘kick in’ only if the energy difference approaches the quality of measurement: $|x - y| \sim \sigma$.

Almost Classical Observables

Collapse with respect to a central observable X is meaningless. This has nothing to do with measurement whatsoever: since the spectral projections $\mathbf{P}(V)$ of a central observable X are central, we see that $\mathbf{C}^*(\rho)(A) = \rho(\sum_i \mathbf{P}(V_i) A \mathbf{P}(V_i)) = \rho(\sum_i \mathbf{P}(V_i) A) = \rho(A)$. Thus $\mathbf{C}^*(\rho) = \rho$ for all states ρ .

This can be generalized for almost classical observables X , i. e. observables for which $d(X, \mathcal{Z})$ is small.

Proposition 24 *Let \mathcal{A} be a von Neumann algebra with centre \mathcal{Z} . Let $A, X \in \mathcal{A}$, X Hermitean. Let $\mathbf{P}(V)$ be the spectral projections of X . Then*

$$\|\mathbf{P}([x, x + \epsilon]) A \mathbf{P}([y, y + \epsilon])\| \leq \frac{\epsilon + 2d(X, \mathcal{Z})}{|x - y|} \|A\|.$$

Proof:

Pretty much the same as that of proposition (23). Under the assumptions above, inequality (3.4) remains valid, as does (3.6) with $\mathbf{M}(B)$ replaced by A . Inequality (3.3) is replaced by

$$\|[X, A]\| \leq 2d(X, \mathcal{Z}) \|A\| \tag{3.7}$$

and (3.5) by

$$\begin{aligned}
\|X_x A - A X_y\| &= \|[X, A] + A(X - X_y) - (X - X_x)A\| \\
&\leq (2d(X, \mathcal{Z}) + \epsilon) \|A\|.
\end{aligned} \tag{3.8}$$

q.e.d.

The same caveat as before applies: if $d(X, \mathcal{Z})$ is non-zero, then off-diagonal blocks close to the diagonal can remain large, so that no bound for $\|\rho - \mathbf{C}^*(\rho)\|$ is obtained.

3.5.4 Generalized State Reduction

In view of proposition (21), it is tempting to speculate that for measurements with good quality ($\sigma \ll 1$), perhaps also $\|(\mathbf{M}^*(\rho))_Y - \mathbf{M}^*(\rho_X)\| \ll 1$ for all ρ . Alas, nature is cruel and hard: consider again the 7th example on page 45. \mathbf{M} (also) measures $X = \begin{pmatrix} 1-\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ with pointer $Y = \mathbb{I} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. One easily calculates $\sigma = \sqrt{\epsilon(1-\epsilon)}$.

But, taking for ρ the spin-down state with density matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, one may figure out (identifying $M_2 \otimes C_2$ with $M_2 \oplus M_2$) that $(\mathbf{M}^*(\rho))_Y$ is represented by the density matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{M}^*(\rho_X)$ by $\begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1-\epsilon \end{pmatrix}$. Consequently $\|(\mathbf{M}^*(\rho))_Y - \mathbf{M}^*(\rho_X)\| = 1 - \epsilon$. Thus, by choosing ϵ small, it is possible to have measurements with $\sigma \ll 1$ yet $\|(\mathbf{M}^*(\rho))_Y - \mathbf{M}^*(\rho_X)\| \approx 1$. In the example above however, the ratio $\frac{\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(X)}{\text{var}_{\mathbf{M}^*(\rho)}(Y)}$ equals 1 for all ϵ . And smallness of this ratio *does* force an approximate reduction, as we will see below. Once again it is not the quality σ ‘an sich’ that regulates reduction, but the quality divided by the typical variations in pointer outcome, cf. proposition (23).

Proposition 25 (Generalized reduction) *Let $X \in \mathcal{A}$, $Y \in \mathcal{B}$ be Hermitean. Let $\mathbf{M} : \mathcal{B} \rightarrow \mathcal{A}$ be an unbiased measurement of X with pointer Y . Then*

$$\|(\mathbf{M}^*(\rho))_Y - \mathbf{M}^*(\rho_X)\| \leq 2 \sqrt{\frac{\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(X)}{\mathbf{M}^*(\rho)(Y^\dagger Y)}} \left(1 + \sqrt{\frac{\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(X)}{\mathbf{M}^*(\rho)(Y^\dagger Y)}} \right).$$

Proof:

Brutally applying the C*-Cauchy-Schwarz inequality would get the job done. That is to say it yields $\|(\mathbf{M}^*(\rho))_Y - \mathbf{M}^*(\rho_X)\| \leq 3\|B\| \frac{\sigma}{\sqrt{\mathbf{M}^*(\rho)(Y^\dagger Y)}}$. Partly because I don’t like the numerator being independent of ρ (allowing $\frac{\sigma^2}{\mathbf{M}^*(\rho)(Y^\dagger Y)}$ to blow up whereas $\frac{\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(X)}{\mathbf{M}^*(\rho)(Y^\dagger Y)}$ is nicely bounded by 1) and partly to keep you from dozing off, we’ll go about it another way¹⁵ for a change. By the GNS-representation (see [K&R][p. 278]), we may assume \mathcal{A} to be an algebra of operators on some Hilbert space \mathcal{H}_ρ , with ρ a vector state ψ_ρ . By the Stinespring theorem (see [Tak, p. 194]), we may assume \mathcal{B} to be an algebra of operators on some Hilbert space \mathcal{H} , and the existence of a contraction $V : \mathcal{H}_\rho \rightarrow \mathcal{H}$ such that \mathbf{M} is of the form $\mathbf{M}(B) = V^\dagger B V$. Then $\mathbf{M}^*\rho$ is a vector state with vector $V\psi_\rho$, since $\mathbf{M}^*(\rho)(B) = \langle \psi_\rho | V^\dagger B V | \psi_\rho \rangle$. In the proof of lemma (13), we have seen that

$$\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(\mathbf{M}(Y)) = \rho(\mathbf{F}(Y, Y)).$$

If we introduce the notation $W \stackrel{\text{def}}{=} \sqrt{(\mathbb{I} - V^\dagger V)}$, we have

$$\rho(\mathbf{F}(Y, Y)) = \langle V\psi_\rho | Y^\dagger W^2 Y | V\psi_\rho \rangle = \|WYV\psi_\rho\|^2.$$

¹⁵I was put on this track by Mădălin Guță, who suggested a simple proof of the C*-Cauchy-Schwarz inequality in the case of completely positive maps.

The rest is hardly exhilarating: for any $B \in \mathcal{B}$,

$$\begin{aligned}
\mathbf{M}^* \rho(Y^\dagger B Y) - \rho(\mathbf{M}(Y)^\dagger \mathbf{M}(B) \mathbf{M}(Y)) &= \\
&= \langle V \psi_\rho | Y^\dagger B Y - Y^\dagger V^\dagger V B V^\dagger V Y | V \psi_\rho \rangle \\
&= \langle V \psi_\rho | Y^\dagger W^2 B Y + Y^\dagger B W^2 Y - Y^\dagger W^2 B W^2 Y | V \psi_\rho \rangle \\
&\leq 2 \|B\| \|Y V \psi_\rho\| \|W Y V \psi_\rho\| + \|B\| \|W Y V \psi_\rho\|^2.
\end{aligned}$$

So, since

$$\mathbf{M}^*(\rho)(Y^\dagger Y) = \rho(\mathbf{M}(Y)^\dagger \mathbf{M}(Y)) + \|W Y V \psi_\rho\|^2$$

we see that

$$\begin{aligned}
&\|(\mathbf{M}^*(\rho))_Y(B) - \mathbf{M}^*(\rho_{\mathbf{M}(Y)})(B)\| = \\
&= \left\| \frac{\mathbf{M}^* \rho(Y^\dagger B Y) \rho(\mathbf{M}(Y)^\dagger \mathbf{M}(Y)) - \rho(\mathbf{M}(Y)^\dagger \mathbf{M}(B) \mathbf{M}(Y)) \mathbf{M}^*(\rho)(Y^\dagger Y)}{\rho(\mathbf{M}(Y)^\dagger \mathbf{M}(Y)) \mathbf{M}^*(\rho)(Y^\dagger Y)} \right\| \\
&= \left\| \frac{\rho(\mathbf{M}(Y)^\dagger \mathbf{M}(Y)) (\mathbf{M}^* \rho(Y^\dagger B Y) - \rho(\mathbf{M}(Y)^\dagger \mathbf{M}(B) \mathbf{M}(Y)))}{\rho(\mathbf{M}(Y)^\dagger \mathbf{M}(Y)) \mathbf{M}^*(\rho)(Y^\dagger Y)} \right. \\
&\quad \left. - \frac{\rho(\mathbf{M}(Y)^\dagger \mathbf{M}(B) \mathbf{M}(Y)) \|W Y V \psi_\rho\|^2}{\rho(\mathbf{M}(Y)^\dagger \mathbf{M}(Y)) \mathbf{M}^*(\rho)(Y^\dagger Y)} \right\| \\
&\leq 2 \|B\| \frac{\|W Y V \psi_\rho\|}{\|Y V \psi_\rho\|} + 2 \|B\| \frac{\|W Y V \psi_\rho\|^2}{\|Y V \psi_\rho\|^2} \\
&= 2 \|B\| \left(\sqrt{\frac{\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(\mathbf{M}(Y))}{\mathbf{M}^*(\rho)(Y^\dagger Y)}} + \frac{\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(\mathbf{M}(Y))}{\mathbf{M}^*(\rho)(Y^\dagger Y)} \right).
\end{aligned}$$

q.e.d.

Since $0 \leq \text{var}_{\mathbf{M}^*(\rho)}(Y) \leq \mathbf{M}^*(\rho)(Y^\dagger Y)$, we may also write down a weaker version, starring the ratio $\frac{\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(X)}{\text{var}_{\mathbf{M}^*(\rho)}(Y)}$ discussed above:

Corollary 25.1 *Let $X \in \mathcal{A}$, $Y \in \mathcal{B}$ be Hermitean. Let $\mathbf{M}: \mathcal{B} \rightarrow \mathcal{A}$ be an unbiased measurement of X with pointer $Y \in \mathcal{B}$ such that $\text{var}_{\mathbf{M}^*(\rho)}(Y) \neq 0$. Then*

$$\|(\mathbf{M}^*(\rho))_Y - \mathbf{M}^*(\rho_X)\| \leq 2 \sqrt{\frac{\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(X)}{\text{var}_{\mathbf{M}^*(\rho)}(Y)}} \left(1 + \sqrt{\frac{\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(X)}{\text{var}_{\mathbf{M}^*(\rho)}(Y)}} \right).$$

If \mathbf{M} is a measurement with outcome 0 or 1, i.e. Y is a projection, then automatically $\text{var}_{\mathbf{M}^*(\rho)}(Y) = p(1-p)$ if p is the probability of measuring outcome 1. From this and $\text{var}_{\mathbf{M}^*(\rho)}(Y) - \text{var}_\rho(X) \leq \sigma^2$, we obtain another corollary.

Corollary 25.2 *Let $X \in \mathcal{A}$ be Hermitean. Let $\mathbf{M}: \mathcal{B} \rightarrow \mathcal{A}$ be an unbiased measurement of X of quality σ which only allows outcomes 0 and 1, i.e. the pointer Y is a projection. Then for all states ρ with probability p of measuring outcome 1:*

$$\|(\mathbf{M}^*(\rho))_Y - \mathbf{M}^*(\rho_X)\| \leq 2 \frac{\sigma}{\sqrt{p(1-p)}} \left(1 + \frac{\sigma}{\sqrt{p(1-p)}} \right).$$

3.6 A Paradox Resolved

Imagine the following thought experiment. The universe is described by the algebra $\mathcal{D} = \mathcal{A} \otimes \mathcal{B}$. An observer $\mathcal{C} \subset \mathcal{B}$ contains two separate pointers Y_1 and Y_2 . (One may think of a computer memory consisting of 2 classical bits, for example.) A perfect measurement $\mathbf{M} : \mathcal{C} \rightarrow \mathcal{A} \otimes \mathcal{B}$ is performed on $X \in \mathcal{A}$ using Y_1 as a pointer. (Information is stored in the first bit.) Since all time-evolution is automorphic, \mathbf{M} must have as dilation some automorphism α of \mathcal{D} . Then there must still be observables $D \in \mathcal{A} \otimes \mathcal{B}$ on which no collapse occurs.

Having learnt the outcome of the first measurement, \mathcal{C} performs a second perfect measurement $\mathbf{N} : \mathcal{C} \rightarrow \mathcal{A} \otimes \mathcal{B}$ but now on D , using Y_2 as pointer. (This information is stored in the second bit.) Comparing information stored on Y_1 with that on Y_2 , \mathcal{C} has solved the riddle of reduction once and for all:

- Either there is a full and objective *reduction* after the first measurement, and the state of \mathcal{D} jumps into an eigenstate of X .
- Or all time evolution is automorphic, and purity on \mathcal{D} is conserved

The difference cannot be seen on observables commuting with Y_1 , but it can be seen on D . (Un?)fortunately, such a crucial experiment is not possible: apply the next proposition to $\alpha \circ \mathbf{N}$.

Proposition 26 *Suppose $Y_1, Y_2 \in \mathcal{B}$ are Hermitean elements such that $[Y_1, Y_2] = 0$. Suppose $\mathbf{M} : \mathcal{C} \rightarrow \mathcal{A} \otimes \mathcal{B}$ is a measurement of $D \in \mathcal{A} \otimes \mathcal{B}$ with pointer Y_2 such that $\mathbf{M}(Y_1) = \mathbb{I} \otimes Y_1$. Then*

$$[D, \mathbb{I} \otimes Y_1] = 0.$$

Proof:

$$\sigma_2 = \|Y_2\|_{\mathbf{M}} = 0, \text{ so } 0 = \mathbf{M}([Y_1, Y_2]) = [\mathbf{M}(Y_1), \mathbf{M}(Y_2)] = [\mathbb{I} \otimes Y_1, D]$$

q.e.d.

It is true that a measurement can be performed on $D \notin (\mathbb{I} \otimes Y_1)'$. But this necessarily erases the information that was gained on X from the pointer Y_1 .

Epilogue

The subject of quantum measurement is particularly susceptible to misunderstanding. I would therefore like to clarify (perhaps superfluously) my view on the so-called ‘measurement problem’ in relation to the interpretation of quantum mechanics ventilated on page 7. The problem of measurement is commonly defined as follows:

How and when do observables take one particular value out of all the possibilities allowed by quantum mechanics?

On page 3 as well as on page 48, I have briefly sketched some of the problems one would have to overcome when answering this question. I do not make any attempt to do so. In my mind, a more relevant question seems to be:

How and when is one particular value of an observable observed by one particular observer, out of all the possibilities allowed by quantum mechanics?

In order to consider this question, one needs a theory with mathematical representatives of both the primitive notion of ‘observable’ and of ‘observer’. Observables are commonly modelled by Hermitean elements. This seems rather sensible to me. But how to model an observer?

In my mind, the key property of any ‘observer’ is that it is able to directly observe a number of observables. I therefore represent an abstract ‘observer’ by the set $\mathcal{C} \subset \mathcal{D}$ of all observables which it can detect directly. Since an observer can construct sums, products and limits from the observed values of observables in \mathcal{C} , it seems plausible that \mathcal{C} is a C^* -algebra. And since simultaneous observation of observables in \mathcal{C} necessarily induces a map of the form discussed in the example following proposition (15), \mathcal{C} may only contain commuting observables. \mathcal{C} is an Abelian C^* -algebra.

One would like to assign a value to each $D \in \mathcal{D}$. This cannot be done in a consistent¹⁶ manner. But with the interpretation on page 7, consistency is only necessary within Abelian algebras. It suffices to have a random generator do the following:

- At time 0, choose an Abelian $\mathcal{C} \subset \mathcal{D}$.
- Assign values to all Hermitean $C \in \mathcal{C}$ in a consistent manner, according to the joint probability measures induced by ρ . These are the values observed by \mathcal{C} at time 0.

¹⁶At least if $\mathcal{D} = \mathcal{B}(\mathcal{H})$ with $\dim(\mathcal{H}) > 2$. See [K&S].

- Repeat this for all possible abelian $\mathcal{C} \subset \mathcal{D}$. Quantum mechanics does not prescribe joint probability distributions for non-commuting observables, so we have some freedom in our choice of random generator. The procedure need not be independent for different observers: there may well be some consistency in their observations. But according to [K&S], full consistency is impossible in general.

Now the question ‘What value of $C \in \mathcal{C}$ is observed by \mathcal{C} at time t ?’ is answered as follows. Look at the Abelian algebra $\alpha_t(\mathcal{C})$ at time 0. The Hermitean observable $\alpha_t(C)$ gets assigned a value in $\mathbf{Spec}(\alpha_t(C)) = \mathbf{Spec}(C)$ with probability distribution $\mathbb{P}_{\rho, \alpha_t(C)} = \mathbb{P}_{\alpha_t^*(\rho), C}$. This is the value of C observed by \mathcal{C} at time t .

This is a deterministic procedure: at time 0, the random generator determines what each observer gets to observe at time t . But observations are not objective. If $A \in \mathcal{C}$ and $A \in \tilde{\mathcal{C}}$, it may well be that A gets different values with \mathcal{C} and $\tilde{\mathcal{C}}$: different observers observe different values of the same observable at the same time. From this, you see that the same observer \mathcal{C} may also observe different values of the same C at different times. The trick is to show that these observations are always consistent with all other observations of the same observer at the same time, and that they allow an observer to store information about the world around it. This can be done entirely within the framework of (quantum) probability theory, without any reference to the nature of the random generator, c.f. page 12, proposition 3, proposition 21 and page 60.

This is my way to interpret these propositions. But I would like to emphasize for one last time that the interpretation above is merely a tool. Any consistent interpretation is just as good as any other. The structure of nature is engraved in mathematics, and interpretations only serve to tie abstract structure to daily experience. This is why I have chosen to be brief in explanation and tedious in calculation. In particular, it explains why the above exposition is muffled away in this epilogue.

Bibliography

- [DAr] G. M. D’Ariano, On the Heisenberg Principle, Namely on the Information-disturbance Trade-off in a Quantum Measurement, *Fortschr. Phys.* **51**, No. 4–5 (2003), 318–330.
- [Dav] E. B. Davies, Quantum Theory of Open Systems, Academic Press, London, 1976.
- [Bel] J. S. Bell, On Wave Packet Reduction in the Coleman-Hepp Model, *Helv. Phys. Acta* **48** (1975), 93–98.
- [Böh] A. Böhm, Quantum Mechanics, Springer Verlag, New York, 1979.
- [B&J] B. H. Bransden and C. J. Joachain, Introduction to Quantum Mechanics, Langman Scientific&technical and Wiley&Sons inc, New York, 1989
- [Coh] Donald Cohn, Measure Theory, Birkhäuser, Boston, 1980.
- [Dir] P. A. M. Dirac, The Principles of Quantum Mechanics, Oxford University Press, London, 1958.
- [Hep] K. Hepp, Quantum Theory of Measurement and Macroscopic Observables, *Helv. Phys. Acta* **45** (1972), 237–248.
- [Hol] A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory, North-Holland Publishing Company, Amsterdam, 1982.
- [Jau] J. M. Jauch, Foundations of Quantum Mechanics, Addison-Wesley, Reading Massachusetts, 1968.
- [K&S] S. Kochen and E. P. Specker, The Problem of Hidden Variables in Quantum Mechanics, *J. Math. Mech.* **17** (1976), 59–87.
- [K&R] R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of Operator Algebras I and II, Academic Press, London, 1983/1986.
- [Kra] K. Kraus, General State Changes in Quantum Theory, *Ann. Phys.* **64** (1971), 311–335.
- [Lan] E. C. Lance, Hilbert C^* -modules: a toolkit for operator algebraists, Cambridge University Press, 1995.

- [Tak] M. Takesaki, Theory of Operator Algebras I, Springer-Verlag, New York, 1979.
- [Maa] J. D. M. Maassen, Quantum Probability, Quantum Information and Quantum Computing, www.math.kun.nl/~medewerkers/maassen, (2004).
- [Neu] J. von Neumann, Mathematische Grundlagen der Quantenmechanik, Springer-Verlag, Berlin, 1932.
- [Wer] R. F. Werner, Quantum Information Theory – an Invitation, *Springer Tracts in Modern Physics* **173** (2001), 14–57. Or alternatively xxx.lanl.gov/abs/quant-ph/0101061>[quant-ph/0101061](http://xxx.lanl.gov/abs/quant-ph/0101061).